

William P. Ziemer

# Weakly Differentiable Functions

*Sobolev Spaces and Functions of  
Bounded Variation*



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