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Nonlinear Functional Analysis and its Applications

III: Variational Methods
and Optimization

Translated by Leo F. Boron

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