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Differential Equations: A Dynamical Systems Approach

Ordinary Differential Equations

With 144 Illustrations

Springer

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Series Preface

Mathematics is playing an ever more important role in the physical and biological sciences, provoking a blurring of boundaries between scientific disciplines and a resurgence of interest in the modern as well as the classical techniques of applied mathematics. This renewal of interest, both in research and teaching, has led to the establishment of the series: *Texts in Applied Mathematics (TAM)* .

The development of new courses is a natural consequence of a high level of excitement on the research frontier as newer techniques, such as numerical and symbolic computer systems, dynamical systems, and chaos, mix with and reinforce the traditional methods of applied mathematics. Thus, the purpose of this textbook series is to meet the current and future needs of these advances and encourage the teaching of new courses.

TAM will publish textbooks suitable for use in advanced undergraduate and beginning graduate courses, and will complement the *Applied Mathematical Sciences (AMS)* series, which will focus on advanced textbooks and research level monographs.

Preface

Consider a first order differential equation of form $x' = f(t, x)$. In elementary courses one frequently gets the impression that such equations can usually be "solved," i.e., that explicit formulas for the solutions (in terms of powers, exponentials, trigonometric functions, and the like) can usually be found. Nothing could be further from the truth. In fact, only very exceptional equations can be explicitly integrated—those that appear in the exercise sections of classical textbooks. For instance, none of the following rather innocent differential equations can be solved by the standard methods:

$$\begin{aligned} x' &= x^2 - t, \\ x' &= \sin(tx), \\ x' &= e^{tx}. \end{aligned}$$

This inability to explicitly solve a differential equation arises even earlier —in ordinary integration. Many functions do not have an antiderivative that can be written in elementary terms, for example:

$$\begin{aligned} &f(t) = e^{-t^2} \quad \text{(for normal probability distribution)}, \\ &f(t) = (t^3 + 1)^{1/2} \quad \text{(elliptic function)}, \\ &f(t) = (\sin t)/t \quad \text{(Fresnel integral)}. \end{aligned}$$

Of course, ordinary integration is the special case of the differential equation $x' = f(t)$. The fact that we cannot easily integrate these functions, however, does *not* mean that the functions above do not have any antiderivatives at all, or that these differential equations do not have solutions.

A proper attitude is the following:

> *Differential equations define functions, and the object of the theory is to develop methods for understanding (describing and computing) these functions.*

For instance, long before the exponential function was defined, the differential equation $x' = rx$ was "studied": it is the equation for the interest x' on a sum of money x, continuously compounded at a rate r. We have records of lending at interest going back to the Babylonians, and the formula that they found for dealing with the problem is the numerical Euler approximation (that we shall introduce in Chapter 3) to the solution of the equation (with the step h becoming the compounding period).

Methods for studying a differential equation fall broadly into two classes: qualitative methods and numerical methods. In a typical problem, both would be used. Qualitative methods yield a general idea of how *all* the solutions behave, enabling one to single out interesting solutions for further study.

Before computer graphics became available in the 1980's, we taught qualitative methods by handsketching direction fields from isoclines. The huge advantage now in exploiting the capabilities of personal computers is that we no longer need to consume huge amounts of time making graphs or tables by hand. The students can be exposed to ever so many more examples, easily, and interactive programs such as MacMath provide ample opportunity and inducement for experimentation any time by student (and instructor).

The origin of this book, and of the programs (which preceded it) was the comment made by a student in 1980: "This equation has no solutions." The equation in question did indeed have solutions, as an immediate consequence of the existence and uniqueness theorem, which the class had been studying the previous month. What the student meant was that there were no solutions that could be written in terms of elementary functions, which were the only ones he believed in. We decided at that time that it should be possible to use computers to *show* students the solutions to a differential equation and how they behave, by using computer graphics and numerical methods to produce pictures for qualitative study. This book and the accompanying programs are the result.

Generally speaking, numerical methods approximate as closely as one wishes a single solution for a particular initial condition. These methods include step-by-step methods (Euler and Runge–Kutta, for instance), power series methods, and perturbation methods (where the given equation is thought of as a small perturbation of some other equation that is better understood, and one then tries to understand how the solution of the known equation is affected by the perturbation).

Qualitative methods, on the other hand, involve graphing the field of slopes, which enables one to draw approximate solutions following the slopes, and to study these solutions all at once. These methods may much more quickly give a rough graph of the behavior of solutions, particularly the long term behavior as t approaches infinity (which in real-world mathematical modeling is usually the most important aspect of a solution). In addition, qualitative techniques have a surprising capacity for yielding specific numerical information, such as location of asymptotes and zeroes. Yet traditional texts have devoted little time to teaching and capitalizing on these techniques. We shall begin by showing how rough graphs of fields of can be used to zero right in on solutions.

In order to accomplish this goal, we must introduce some new terminology right at the beginning, in Chapter 1. The descriptive terms "fence," "funnel," and "antifunnel" serve to label simple phenomena that have exceedingly useful properties not exploited in traditional treatments of differential equations. These simple new ideas provide a means of formalizing the observations made by any person *familiar* with differential equations, and they provide enormous payoff throughout this text. They give simple, direct, noniterative proofs of the important theorems: an example is the Sturm comparison and oscillation theorem, for which fences and funnels quickly lead to broad understanding of all of Sturm–Liouville theory. Actually, although the words like fences and funnels are new, the notions have long been found under the umbrella of differential inequalities. However, these notions traditionally appeared without any drawings, and were not mentioned in elementary texts.

Fences and funnels also yield hard quantitative results. For example, with the fences of Chapter 1 we can often prove that certain solutions to a given differential equation have vertical asymptotes, and then calculate, to as many decimal places as desired, the location of the asymptote for the solution with a particular initial condition. Later in Part III, we use fences forming an antifunnel to easily calculate, with considerable accuracy, the roots of Bessel functions. All of these fruits are readily obtained from the introduction of just these three well-chosen words.

We solve traditionally and explicitly few types of first order equations—linear, separable, and exact—in Chapter 2. These are by far the most useful classical methods, and they will provide all the explicit solutions we desire.

Chapter 4 contains another vital aspect to our approach that is not provided in popular differential equations texts: a Fundamental Inequality (expanding on the version given by Jean Dieudonné in *Calcul Infinitésimal;* see the References). This Fundamental Inequality gives, by a constructive proof, existence and uniqueness of solutions *and* provides error estimates. It solidly grounds the numerical methods introduced in Chapter 3, where a fresh and practical approach is given to error estimation.

Part I closes with Chapter 5 on iteration, usually considered as an entirely different discipline from differential equations. However, as another type of dynamical system, the subject of iteration sheds direct light on how stepsize determines intervals of stability for approximate solutions to a differential equation, and to gain understanding (through Poincaré mapping) of solutions to periodic differential equations, especially with respect to bifurcation behavior.

In subsequent volumes, Parts II, III and IV, as we add levels of complexity, we provide simplicity and continuity by cycling the same concepts introduced in Part I. Part II begins with Chapter 6, where we extend $x' = f(t,x)$ to the multivariate vector version $\mathbf{x}' = \mathbf{f}(t,\mathbf{x})$. This is also the form to which a higher order differential equation in a single variable can be reduced. Chapter 7 introduces linear differential equations of the form $\mathbf{x}' = A\mathbf{x}$, where eigenvalues and eigenvectors accomplish transformation of the vector equation into a set of decoupled single variable first order equations. Chapters 8 and 9 deal with nonlinear differential equations and bifurcation behavior.

In Part III, Chapters 10 and 11 discuss applications to electrical circuits and mechanics respectively. Chapter 12 deals with linear differential equations with nonconstant coefficients, $\mathbf{x}' = A(t)\mathbf{x} + \mathbf{q}(t)$, and includes Sturm–Liouville theory and the theory of ordinary singular points. Finally, Part III again fills out the dynamical systems picture, and closes with Chapter 13 on iteration in two dimensions.

In Part IV, partial differential equations and Fourier series are introduced as an infinite-dimensional extension of the same eigenvalue and eigenvector concept that suffuses Part II. The remaining chapters of the text continue to apply the same few concepts to all the famous differential equations and to many applications, yielding over and over again hard quantitative results. For example, such a calculation instantly yields, to one part in a thousand, the location of the seventh zero of the Bessel function J_0; the argument is based simply on the original concepts of fence, funnel, and antifunnel.

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The enormous job of providing the final illustrations has been shared by the authors, and Jeesue Kim, Maria Korolov, Scott Mankowitz, Katrina Thomas, and Thomas Yan (whose programming skill made possible the high quality computer output). Homer Smith of ArtMatrix made the intricate pictures of Mandelbrot and Julia sets for Chapter 5.

Anne Noonburg gets credit for the vast bulk of work in providing solutions to selected exercises. However, the authors take complete responsibility for any imperfections that occur there or elsewhere in the text.

Others who have contributed considerably behind the scenes on the more mechanical aspects at key moments include Karen Denker, Mary Duclos, Fumi Hsu, Rosemary MacKay, Jane Staller, and Frederick Yang.

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Finally, we are grateful to the editors and production staff at Springer-Verlag for their assistance, good ideas, and patience in dealing with a complicated combination of text and programs.

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Ways to Use This Book

There are many different ways you might use this book. John Hubbard uses much of it (without too much of Chapter 5) in a junior–senior level course in applicable mathematics, followed by *Part II: Higher Dimensional Differential Equations*, *Part III: Higher Dimensional Differential Equations Continued*, and *Part IV: Partial Differential Equations*. In each term different chapters are emphasized and others become optional.

Most instructors prefer smaller chunks. A good single-semester course could be made from Chapters 1 and 2, then a lighter treatment of Chapter 3; Chapter 4 and most of Chapter 5 could be optional. Chapters 6, 7, and 8 from Part II comprise the core of higher dimensional treatments. Chapter 7 requires background in linear algebra (provided in the Appendix to Part II), but this is not difficult in the two- and three-dimensional cases. Chapter 8 is important for showing that a very great deal can be done today with nonlinear differential equations.

This series of books has been written to take advantage of computer graphics. We've developed a software package for the Macintosh computer called *MacMath* (which includes *Analyzer, DiffEq, Num Meths, Cascade, 1D Periodic Equations*) and refer to it throughout the text.

Although they are not absolutely essential, we urge the use of computers in working through this text. It need not be precisely with the *MacMath* programs. With IBM/DOS computers, readers can, for example, use *Phaser* by Huseyn Koçak or *MultiMath* by Jens Ole Bach. There are many other options. The chapter on numerical methods has been handled very successfully with a spreadsheet program like *Excel.*

Because so much of this material is a new approach for instructors as well as students, we include a set of solutions to selected exercises, as a guide to making the most of the text.

Contents of Part I

Ordinary Differential Equations
The One-Dimensional Theory $x' = f(t, x)$

Contents of Part II

Contents of Part III

Contents of Part IV