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Preface

It is just over twenty years since the *The Finite Element Method in Structural and Continuum Mechanics* was first published. This book, which was the first dealing with the finite element method, provided the base from which many further developments occurred. The expanding research and field of application of finite elements led to the second edition in 1971 and the third in 1977. The size of each of these volumes expanded geometrically (from 272 pages in 1967, 521 pages in 1971, to 787 pages in 1977). This was necessary to do justice to a rapidly expanding field of professional application and research. Even so, much filtering of the contents of the third edition was necessary to keep it within reasonable bounds.

As in essence the matters published in the third edition are still valid today and this forms a useful and widely used text and reference book, we have decided to publish an *expanded version* in two volumes. These will retain as far as possible the contents of the third edition and add or reinterpret matters which today have become of added importance.

The division of the contents between the two volumes follows the lines of instruction for which the book can serve either by self study, as we anticipate the book to be used widely by practising engineers, or in university courses for engineers and physicists. The first volume is thus devoted to the basic finite element approximation concepts and to simple linear, static, computations which even today provide the major part of the finite element usage.

We have relegated to the second volume all problems of dynamics, of non-linear solution techniques, and, indeed, the linear problems of plates and shells which introduce special difficulties and where optimal techniques are yet debated.

The contents of the first volume are slightly rearranged from those of