### About the Author

ed in Poland. He real lege, London. In 194 Southwell – pioneer awarded his PhD in 1 as a consulting engited Edinburgh before be Engineering at North USA. He returned to of the Civil Enginee Wales, Swansea. Hi cal Methods in Engand most active resident research and

In 1965 he was awa sity of London and nized by the award these in Lisbon in Sweden (Gottenb (Dalian), with Pola

In 1979, he was eleciety for his achie and for his other other honours into of the United Stat 1981, Foreign Me in 1985, and the medal by the We many — only the honour. In 1986 the President of tional Mechanics British Empire band was preser

Professor Tayl University of Ca Professor of Cit the department a half years.

During his time Division of Struics for three ye

Professor Tayl analysis since batical at Swa Professor Zie awarded the Swansea, in

# Contents

	xiii	
Preface	XV	
Acknowledgements		
List of Symbols	xvii	
To STANDARD DISCRETE SYST	ты 1	
CHAPTER 1 SOME PRELIMINARIES: THE STANDARD DISCRETE SYST	1	
1.1 Introduction	3	
1.2 The structural element and system	9	
1.3 Assembly and analysis of a structure	11	
1.4 The boundary conditions	12	
1.5 Electrical and fluid networks	14	
1.6 The general pattern	16	
1.7 The standard discrete system	17	
1.8 Transformation of coordinates	17	
Chapter 2 Finite Elements of an Elastic Continuum-		
DISPLACEMENT APPROACH	21	
2.1 Introduction	21	
2.1 Introduction  2.2 Direct formulation of finite element characteristics	23	
2.2 Direct formulation of finite electrons of mine electrons of finite electrons of fi		
2.3 Generalization to the whole region	30	
concept abandoned  2.4 Displacement approach as a minimization of total		
2.4 Displacement approach as a minimization of total	32	
potential energy	35	
2.5 Convergence criteria	37	
2.6 Discretization error and convergence rate		
2.7 Displacement functions with discontinuity between	38	
elements—non-conforming elements and the patch test	39	
2.8 Bound on strain energy in a displacement formulation	40	
2.9 Direct minimization	41	
2.10 An example	1.2	
vii		

the Authors

ai
nd
AI -
hi
su
gh
ring
e re
civil
Swa.
thoc
st a
esea
he v
Lond
by the
n Lis
n (G
n), wit
9, he \
or his
r his o
honou
United
Foreig
35, and
Lhutha

ıl Mecha
sh Empir
was pre
essor Ta
ersity of
essor of
departme
alf vears.

ing his tin ision of Str for three
ofessor Tay alysis since
tical at Swa
ofessor Zie rarded the h
vansea, in

THE FINITE ELEMENT METHOD	
	45
CHAPTER 3 PLANE STRESS AND PLANE STRAIN	45
2.1 Introduction	46
1 actorictics	56
Tamber—An assessment of	59
	72
CHAPTER 4 AXISYMMETRIC STRESS ANALYSIS	72
4.1 Introduction	73
Floment characteristics	81
"Il-atrotive examples	81
Description applications	85
4.4 Practical applications 4.5 Non-symmetrical loading	86
· · · · · · · · · · · · · · · · · · ·	
4.6 Axisymmetry plants	89
CHAPTER 5 THREE-DIMENSIONAL STRESS ANALYSIS	89
	90
5.1 Introduction 5.2 Tetrahedral element characteristics	95
'i alaments Willi Cigit in	98
5.3 Composite elements was 5.4 Examples and concluding remarks	,,
5.4 Examples and concluding	
CHAPTER 6 TENSOR-INDICIAL NOTATION IN THE APPROXIMATION	103
CHAPTER 6 TENSOR-INDICIAL NOTATION OF ELASTICITY PROBLEMS	103
	103
6.1 Introduction	103
6.2 Indicial notation	
6.2 Indicial notation 6.3 Derivatives and tensorial relations	107
6.4 Flastic materials and lillite clothens	
'HIPPARCHICAL' ELEMENT SHAPE	
Chapter 7 'Standard' and Hierarchites $C_0$ Functions: Some General Families of $C_0$	410
FUNCTIONS: SOME GENERAL I AMA	110
CONTINUITY	110
7.1 Introduction	112
7.1 Introduction 7.2 Standard and hierarchical concepts 7.3 Rectangular elements—some preliminary considerations	114
D tangular elements—some provider	113
	11
7.4 Completeness of polynomias 7.5 Rectangular elements—Lagrange family 7.5 rectangular elements—'serendinity' family	12
7.5 Rectangular elements—'serendipity' family 7.6 Rectangular elements—'serendipity' family—	
-tion of internal variables	12
hotructures	12
1 1-mont family	13
7.8 Triangular elements 7.9 Linear elements  'serendipity' family	1
	1
7.10 Rectangular prisms—Lagrange family 7.11 Rectangular prisms—Lagrange family	1
7.11 Rectangular products 7.12 Tetrahedral elements	
7.12 Tetraneural olds	

CONTENTS	ix
CONTENTS	139
the shape dimensional elements	139
7.13 Other simple three-university of the 7.14 Hierarchic polynomials in one dimension 7.14 Hierarchic polynomials hierarchic, elements of the	
7.15 Two- and three-dimensional, and	142
to mala or Drick Lype	144
	146
7.16 Triangle and tetrahedron failing 7.17 Global and local finite element approximation 7.18 Improvement of conditioning with hierarchic forms	147
CHAPTER 8 MAPPED ELEMENTS AND NUMERICAL INTEGRATION—	150
CHAPTER 8 MAPPED ELEMENTS AND TOTAL STREET S	150 150
	150
<ul><li>8.1 Introduction</li><li>8.2 Use of 'shape functions' in establishment of coordinate</li></ul>	153
C ott one	158
	159
Variation of the unknown functions $\xi$ . Variation of the unknown function $\xi$ . Variation of the unknown function $\xi$ . Variation of the unknown function $\xi$ .	
85 Evaluation of element matrices (France)	161
ζ coordinates)	164
<ul> <li>ζ coordinates)</li> <li>8.6 Element matrices. Area and volume coordinates</li> <li>8.7 Convergence of elements in curvilinear coordinates</li> <li>8.8 Convergence of elements in curvilinear coordinates</li> </ul>	166
8.7 Convergence of elements in curvational	171 174
8.8 Numerical integration of right prism regions	174
8.9 Numerical integration—rectangular of figure positions 8.10 Numerical integration—triangular or tetrahedral regions	177
8.10 Numerical integration 8.11 Required order of numerical integration 8.11 Required order of numerical integration 8.11 Required order of numerical integration	177
8.11 Required order of numerical integration 8.12 Generation of finite element meshes by mapping. Blending	181
	183
functions 8.13 Infinite domains and infinite elements 8.13 Infinite domains and infinite elements	189
8.13 Infinite domains and infinite elements 8.14 Singular elements by mapping for fracture mechanics, etc. 8.14 Singular elements by mapping for fracture mechanics, etc.	
8.14 Singular elements by mapping for fractare 8.15 A computational advantage of numerically integrated	191
finite elements	192
finite elements 8.16 Some practical examples of two-dimensional stress analysis	195
0 17 Three-dimensional stross distrib	200
8.17 Timec-dimensional 8.18 Symmetry and repeatability	
CHAPTER 9 GENERALIZATION OF THE FINITE ELEMENT	
CONCEPTS, GALERKIN-WEIGHTED TESTS	206
VARIATIONAL APPROACHES	206
	1
9.2 Integral or 'weak' statements equivalent	210
equations equation—forced and	
equations 9.3 Weak form of the heat conduction equation—forced and	212
natural boundary conditions	

the Author

sor Zienkiew
oland. He rea
.ondon. In 19
well – pioneer
ed his PhD in
onsulting eng
urgh before to
eering at Nort.
He returned to
e Civil Engine
s, Swansea. I
Methods in Er
most active re
ent research ar

965 he was av of London and ed by the awa ese in Lisbon in veden (Gotter valian), with Po

interpretation in 1979, he was liety for his action of the United Straight in 1981, Foreign in 1985, and the many — only honour. In 198 the President tional Mechan British Empire and was president in 1986.

Professor of the departm a half years During his Division of

Professor T University of

> Professor analysis si batical at s Professor awarded Swansea

ics for thre

х	THE FINITE ELEMENT METHOD	
9.4	Approximation to integral formulations: the weighted	
80.70	residual—Galerkin method	214
9.5	Examples	215
9.6	Virtual work as the 'weak form' of equilibrium	
	equations for analysis of solids or fluids	223
9.7	Partial discretization	225
	Convergence	228
9.9	What are 'variational principles'?	230
9.10	'Natural' variational principles and their relation to	
	governing differential equations	233
9.11	Establishment of natural variational principles for linear,	
	self-adjoint differential equations	238
9.12	Maximum, minimum, or a saddle point?	242
9.13	Constrained variational principles. Lagrange multipliers	
	and adjoint functions	243
9.14	Constrained variational principles. Penalty functions and the	he least
	square method	249
9.15	Concluding remarks—finite differences and boundary	
	methods	256
0	TER 10 STEADY-STATE FIELD PROBLEMS—HEAT	
CHAR	PTER 10 STEADY-STATE FIELD PROBLEMS—HEAT CONDUCTION, ELECTRIC AND MAGNETIC POTENTIAL	
		260
10.1	FLUID FLOW, ETC. Introduction	260
10.1	The general quasi-harmonic equation	261
10.2	Finite element discretization	263
10.3	Some economic specializations	264
10.4	Examples—an assessment of accuracy	267
10.5	Some practical applications	271
10.6	Concluding remarks	287
10.7	Concluding remarks	207
CHAI	PTER 11 THE PATCH TEST, REDUCED INTEGRATION,	
	AND NON-CONFORMING ELEMENTS	290
11.1	Introduction	290
11.2	Convergence requirements	292
11.3	The simple patch test (forms A and B)—a necessary	
	condition for convergence	293
11.4	Generalized patch test (test C) and the single-element test	295
11.5	Higher order patch tests	297
11.6	Application of the patch test to plane elasticity	
		000

elements with 'standard' and 'reduced' quadrature

11.7 Application of the patch test to an incompatible element

298

304

	CONTENTS	X1		
11.8	Generation of incompatible shape functions which			
	satisfy the patch test	308		
11.9	in the second se			
11.10	Higher order patch test example—robustness	312		
	Closure	316		
Снар	TER 12 MIXED FORMULATION AND CONSTRAINTS—			
	COMPLETE FIELD METHODS	319		
12.1	Introduction	319		
12.2	Discretization of mixed forms—some general remarks	321		
12.3	Stability of mixed approximation. The patch test	324		
12.4	Mixed formulation in elasticity	327		
12.5	Incompressible (or nearly incompressible) elasticity	334		
12.6	Stress smoothing/optional sampling	345		
12.7	Reduced and selective integration and its equivalence to			
	penalized mixed problems	351		
12.8	A simple iterative solution process for mixed problems	357		
12.9	Complementary forms with direct constraint	364		
12.10	Concluding remarks—mixed formulation or a test of			
	element 'robustness'	367		
Снар	TER 13 MIXED FORMULATION AND CONSTRAINTS—			
	INCOMPLETE (HYBRID) FIELD METHODS	373		
13.1		373		
13.2	Interface traction link of two (or more) irreducible			
	form subdomains	373		
13.3	Interface traction link of two or more mixed form			
	subdomains	376		
13.4	Interface displacement 'frame'	378		
13.5	Linking of boundary (or Trefftz)-type solution by 'frame'			
	of specified displacements	389		
13.6	Subdomains with 'standard' elements and global functions	395		
13.7	Concluding remarks	395		
Снар	TER 14 ERROR ESTIMATES AND ADAPTIVE FINITE ELEMENT			
	REFINEMENT	398		
14.1	Introduction	398		
14.2	Error norms and convergence rates	400		
14.3	Error estimates—a simple and effective procedure for h			
	refinement	407		
14.4	The h refinement process—adaptivity	422		
14.5	Error estimates for hierarchic formulations. A basis for			
ì	p-adaptive refinement	426		
14.6	Concluding remarks	429		

### About th

Professo ed in Pola lege, Lon Southwel awarded as a cons Edinburg Engineer USA. He of the Ci Wales, S ical Metl and mos

> In 1965 sity of I nized b these in Swede (Dalian

ment re

In 1979 ciety f and fo other of the 1981, in 1988 meda many honor the Pitional Britis and of the Pitional Britis and other for the pitional Britis and other forms.

Profe University Profe the d a hal

Profession Profession Swa

#### THE FINITE ELEMENT METHOD

****		THE THIRD ELEMENT METHOD	
Сна	PTER 15	COMPUTER PROCEDURES FOR FINITE ELEMENT	
		Analysis	436
15.1	Introd	uction	436
15.2	Data i	nput module	439
15.3	User in	nstructions for computer program	450
15.4		on of finite element problems—the macro	
		mming language	459
15.5	Comp	utation of finite element solution modules	466
15.6	Solution	on of simultaneous, linear algebraic equations	479
15.7	Extens	sions and modification to the computer program	492
15.8	Listing	g of finite element computer program	495
Appe	NDIX 1	Matrix algebra	584
APPE	NDIX 2	Basic equations of displacement analysis (Chapter 2)	590
APPE	NDIX 3	Some integration formulae for a triangle	591
APPENDIX 4		Some integration formulae for a tetrahedron	592
APPE	NDIX 5	Some vector algebra	594
APPE	NDIX 6	Integration by parts in two or three dimensions	
		(Green's theorem)	599
APPE	NDIX 7	Solutions exact at nodes	601
APPE	NDIX 8	Matrix diagonalization or lumping	605
Auti	OR IND	EX	611
Subji	Subject Index		616

## Preface

It is just over twenty years since the *The Finite Element Method in Structural and Continuum Mechanics* was first published. This book, which was the first dealing with the finite element method, provided the base from which many further developments occurred. The expanding research and field of application of finite elements led to the second edition in 1971 and the third in 1977. The size of each of these volumes expanded geometrically (from 272 pages in 1967, 521 pages in 1971, to 787 pages in 1977). This was necessary to do justice to a rapidly expanding field of professional application and research. Even so, much filtering of the contents of the third edition was necessary to keep it within reasonable bounds

As in essence the matters published in the third edition are still valid today and this forms a useful and widely used text and reference book, we have decided to publish an *expanded version* in two volumes. These will retain as far as possible the contents of the third edition and add or reinterpret matters which today have become of added importance.

The division of the contents between the two volumes follows the lines of instruction for which the book can serve either by self study, as we anticipate the book to be used widely by practising engineers, or in university courses for engineers and physicists. The first volume is thus devoted to the basic finite element approximation concepts and to simple linear, static, computations which even today provide the major part of the finite element usage.

We have relegated to the second volume all problems of dynamics, of non-linear solution techniques, and, indeed, the linear problems of plates and shells which introduce special difficulties and where optimal techniques are yet debated.

The contents of the first volume are slightly rearranged from those of