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Differentiation of Integrals in R^n

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PREFACE

The work presented here deals with the local aspect of the differentiation theory of integrals. This theory takes its origin in the wellknown theorem of Lebesgue [1910]: Let f be a real function in $L^1(R^n)$. Then, for almost every $x \in R^n$ we have, for every sequence of open Euclidean balls $B(x,r_k)$ centered at x such that $r_k \to 0$,

$$\lim (1/|B(x,r_k)|) \int_{B(x,r_k)} f(y)dy = f(x) \quad \text{as} \quad k \to \infty.$$

One could think that the fact that one takes here the limit of the means over Euclidean balls instead of taking them over other type of sets contracting to the point x might well be irrelevant. It was not until about 1927 that H. Bohr exhibited an example, first published by Carathéodory [1927], showing that intervals in R^2 (i.e. rectangles with sides parallel to the axes) behave much worse than cubic intervals or circles with regard to a covering property (Vitali's lemma) that was fundamental for the result of Lebesgue. So it became a challenging problem to find out whether the replacement of Euclidean balls by intervals centered at the point x in the Lebesgue theorem would lead to a true statement or not. The first result in this direction was the so-called strong density theorem, first proved by Saks [1933], stating that if the function f is the characteristic function of a measurable set, then Ecuclidean balls can be replaced by intervals. Later on Zygmund [1934] showed that this can also be done if f is in any space $L^p(R^n)$, with $1 < p \leq \infty$, and a year later Jessen, Marcinkiewicz and Zygmund [1935] proved that the same is valid if f is in $L(1+\log^+ L)^{n-1}(R^n)$. On the other hand Saks [1934] proved that there exists a function g in $L(R^n)$ such that the Lebesgue statement is false for g if one take intervals instead of balls. The Fundamenta Mathematicae of those years, which still remains one of the main sources of information for the theory of differentiation bears testimony to the interest of many outstanding mathematicians for this subject.

One of the important products of such activity was the surprising result that, if in the Lebesgue theorem one tries to replace circles by rectangles centered at the point x then the statement is not any more true in general even if f is assumed to be the characteristic function of a measurable set. This was first observed by Zygmund as a byproduct of the construction by Nikodym [1927] of a certain paradoxical set.

Such findings prompted others to try to consider more general situations and to give some characterization of those families of sets that, like the Euclidean balls or the intervals, would permit a differentiation theorem similar to that of Lebesgue. The first attempts in this direction were the fundamental paper of Busemann and Feller [1934], giving such a characterization by means of a certain "halo" condition, and the paper by de Possel [1936], offering one in terms of a covering property.

In this way there arose the theory of differentiation, which as we shall have occasion to show, still presents many challenging open problems and has very interesting connections with other branches of analysis. In the present work I have tried to focus on some of the more fundamental aspects of the differentiation theory of integrals in R^n. In this context the theory can be presented very concretely and with a minimal amount of terminology. Many interesting open problems, whose solution will probably lead to a better undertanding of basic structures in analysis, can be stated in a way simple enough to be inmediately understood by those who just know what is a Lebesgue measurable function defined on R^2.

The differentiation theory we shall present here appears as an interaction between covering properties of families of sets in R^n, differentiation properties similar to that of the Lebesgue theorem, and estimations for an adequate extension of the wellknown maximal operator of Hardy and Littlewood. The whole book is a commentary on these three main subjects.

Chapter I is devoted to the main covering theorems that are used in the subject. Chapter II introduces the notions of a differentiation basis and of the maximal operator associated to it, and offers certain basic methods in order to obtain several useful estimations for this operator. Chapter III shows how closely related are the properties of the maximal operator and the differentiation properties of a basis. Chapters IV, V and VI explore some properties of several examples of differentiation bases, the basis of intervals, that of rectangles, and of some special sets (convex sets and unbounded star-shaped sets). Chapter VII is devoted to the possibility of obtaining covering properties starting from differentiation properties of a basis. Finally Chapter VIII contains some considerations about a particular problem in which the author has been interested.

Each chapter is divided in sections. I have tried to offer in the main body of each section just the relevant result that has been the source of inspiration for many other further developments. In the remarks at the end of each section I give information, often rather detailed, about some extensions of the theory, without trying at all to be exhaustive. In the theory we present there are still many open problems. I have stated some of them, almost always in the remarks at the end of each section. A list of them is given at the end. Some of these problems might be easy to solve, but some others seem to be rather difficult and will perhaps require fresh ideas and new techniques in our field. I hope that some of the readers will be stimulated by such problems and so the theory will be enriched with their effort. I would certainly be very grateful for any light on these problems I might receive from them. I am very happy to say that after the first version of these notes was written, in December 1.974, some of the problems proposed in them have been solved and some others have been substantially illuminated. In the appendices at the end of this work, written by A. Córdoba, R. Fefferman and R. Moriyón one can see some of the progress that has been made. I wish to thank them for having permitted me to include in these notes their results, that will be of great value for those interested in the field. Also very recently C. Hayes has solved in a very general setting the problem proposed in page 165.

I wish to thank, first of all, Prof. Antoni Zygmund for the encouragement I have received from him to write this work and for many helpful discussions on the subject. The assistance and helpful criticism of my colleagues at the University of Madrid has been invaluable. I owe particular gratitude to C. Aparicio, M.T. Carrillo, J. López, M.T. Menárguez, R. Moriyón, I. Peral, B. Rubio and M. Walias for many stimulating hours we have spent discussing the topics treated here. I also wish to thank A.M. Bruckner, C. Hayes and G. V. Welland for having read the first version of these notes and for their very helpful suggestions. Paloma Rodríguez, Isi Vázquez and Pablo Mz. Alirangües were in charge of typing and preparing these notes for publication. I thank them very much for their fine job.

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For a point x in R^n, $|x|$ means the Euclidean norm of x, i.e. if x = $= (x_1, x_2, \ldots, x_n)$, then

$$|x| = (\sum_{i=1}^{n} x_i^2)^{1/2}.$$

For a set A in R^n, $|A|_e$ means the exterior Lebesgue measure of A, $\delta(A)$ the (Euclidean) diameter of A, ∂A the boundary of A. If A is measurable, $|A|$ denotes its measure, and A' denotes the complement of A.

For a sequence $\{A_k\}$ of subsets of R^n and a point $x \in R^n$, $A_k \to x$ ("A_k contracts to x") means that $x \in A_k$ for each k and $\delta(A_k) \to 0$.

For a sequence $\{r_k\}$ of real numbers and $a \in R$, $r_k \uparrow a$ ($r_k \downarrow a$) means that r_k converges increasingly (decreasingly) to a.

For a family $\mathcal{A}$ of sets of R^n, $\bigcup \{A : A \in \mathcal{A}\}$ means the set of points of R^n belonging to some set of the family $\mathcal{A}$.