

Serge Lang

# Calculus of Several Variables

Third Edition

With 298 Illustrations



Springer

Serge Lang  
Department of Mathematics  
Yale University  
New Haven, CT 06520  
USA

*Editorial Board*

S. Axler Mathematics Department San Francisco State University San Francisco, CA 94132 USA	F.W. Gehring Mathematics Department East Hall University of Michigan Ann Arbor, MI 48109 USA	K.A. Ribet Department of Mathematics University of California at Berkeley Berkeley, CA 94720-3840 USA
--	---	---

---

Mathematics Subject Classification (1991): 26-01, 26Bxx

---

Library of Congress Cataloging in Publication Data

Lang, Serge

Calculus of several variables.  
(Undergraduate texts in mathematics)  
Includes index.  
1. Calculus. 2. Functions of several real  
variables. I. Title. II. Series.  
QA303.L256 1987 515.8'4 86-17724

Previous editions of this book were published by Addison-Wesley, Reading,  
Massachusetts, 1973 and 1979.

Printed on acid-free paper.

© 1987 Springer Science+Business Media New York  
Originally published by Springer-Verlag New York, Inc. in 1987  
Softcover reprint of the hardcover 3rd edition 1987

All rights reserved. This work may not be translated or copied in whole or in part without  
the written permission of the publisher Springer Science+Business Media, LLC, except  
for brief excerpts in connection with reviews or scholarly analysis. Use in connection  
with any form of information storage and retrieval, electronic adaptation, computer  
software, or by similar or dissimilar methodology now known or hereafter developed is  
forbidden.

Typeset by Composition House Ltd., Salisbury, England.

9 8 7 6

ISBN 978-1-4612-7001-0 ISBN 978-1-4612-1068-9 (eBook)  
DOI 10.1007/978-1-4612-1068-9

SPIN 10792691

# Contents

## PART ONE

<b>Basic Material . . . . .</b>	1
---------------------------------	---

## CHAPTER I

<b>Vectors . . . . .</b>	3
§1. Definition of Points in Space . . . . .	3
§2. Located Vectors . . . . .	11
§3. Scalar Product . . . . .	14
§4. The Norm of a Vector . . . . .	17
§5. Parametric Lines . . . . .	32
§6. Planes . . . . .	36
§7. The Cross Product. . . . .	44

## CHAPTER II

<b>Differentiation of Vectors . . . . .</b>	49
§1. Derivative . . . . .	49
§2. Length of Curves. . . . .	62

## CHAPTER III

<b>Functions of Several Variables . . . . .</b>	66
§1. Graphs and Level Curves . . . . .	66
§2. Partial Derivatives . . . . .	70
§3. Differentiability and Gradient . . . . .	77
§4. Repeated Partial Derivatives . . . . .	82

## CHAPTER IV

<b>The Chain Rule and the Gradient . . . . .</b>	87
§1. The Chain Rule . . . . .	87
§2. Tangent Plane . . . . .	92
§3. Directional Derivative . . . . .	99
§4. Functions Depending only on the Distance from the Origin . . . . .	103
§5. The Law of Conservation of Energy . . . . .	111
§6. Further Technique in Partial Differentiation . . . . .	114

## PART TWO

<b>Maxima, Minima, and Taylor's Formula . . . . .</b>	121
---	-----

## CHAPTER V

<b>Maximum and Minimum . . . . .</b>	123
§1. Critical Points . . . . .	123
§2. Boundary Points . . . . .	126
§3. Lagrange Multipliers . . . . .	135

## CHAPTER VI

<b>Higher Derivatives . . . . .</b>	143
§1. The First Two Terms in Taylor's Formula . . . . .	143
§2. The Quadratic Term at Critical Points . . . . .	149
§3. Algebraic Study of a Quadratic Form . . . . .	155
§4. Partial Differential Operators . . . . .	162
§5. The General Expression for Taylor's Formula . . . . .	170
Appendix. Taylor's Formula in One Variable . . . . .	176

**Note.** Chapter IX on Double Integrals is self contained, and could be covered here.

## PART THREE

<b>Curve Integrals and Double Integrals . . . . .</b>	181
---	-----

## CHAPTER VII

<b>Potential Functions . . . . .</b>	183
§1. Existence and Uniqueness of Potential Functions . . . . .	184
§2. Local Existence of Potential Functions . . . . .	188
§3. An Important Special Vector Field . . . . .	194
§4. Differentiating Under the Integral . . . . .	198
§5. Proof of the Local Existence Theorem . . . . .	201

## CHAPTER VIII

<b>Curve Integrals . . . . .</b>	206
----------------------------------	-----

§1. Definition and Evaluation of Curve Integrals. . . . .	207
§2. The Reverse Path . . . . .	217

§3. Curve Integrals When the Vector Field Has a Potential Function . . . . .	220
§4. Dependence of the Integral on the Path. . . . .	228

## CHAPTER IX

<b>Double Integrals</b> . . . . .	233
§1. Double Integrals . . . . .	233
§2. Repeated Integrals . . . . .	242
§3. Polar Coordinates . . . . .	252

## CHAPTER X

<b>Green's Theorem</b> . . . . .	269
§1. The Standard Version . . . . .	269
§2. The Divergence and the Rotation of a Vector Field . . . . .	280

## PART FOUR

<b>Triple and Surface Integrals</b> . . . . .	291
---	-----

## CHAPTER XI

<b>Triple Integrals</b> . . . . .	293
§1. Triple Integrals . . . . .	293
§2. Cylindrical and Spherical Coordinates . . . . .	298
§3. Center of Mass . . . . .	313

## CHAPTER XII

<b>Surface Integrals</b> . . . . .	318
§1. Parametrization, Tangent Plane, and Normal Vector . . . . .	318
§2. Surface Area . . . . .	325
§3. Surface Integrals . . . . .	333
§4. Curl and Divergence of a Vector Field . . . . .	342
§5. Divergence Theorem in 3-Space . . . . .	345
§6. Stokes' Theorem . . . . .	355

## PART FIVE

<b>Mappings, Inverse Mappings, and Change of Variables Formula</b> .	365
--	-----

## CHAPTER XIII

<b>Matrices</b> . . . . .	367
§1. Matrices . . . . .	367
§2. Multiplication of Matrices . . . . .	372

<b>CHAPTER XIV</b>	
<b>Linear Mappings . . . . .</b>	<b>385</b>
§1. Mappings . . . . .	385
§2. Linear Mappings . . . . .	392
§3. Geometric Applications . . . . .	398
§4. Composition and Inverse of Mappings. . . . .	404
<b>CHAPTER XV</b>	
<b>Determinants . . . . .</b>	<b>412</b>
§1. Determinants of Order 2 . . . . .	412
§2. Determinants of Order 3 . . . . .	416
§3. Additional Properties of Determinants . . . . .	420
§4. Independence of Vectors . . . . .	428
§5. Determinant of a Product . . . . .	430
§6. Inverse of a Matrix . . . . .	431
<b>CHAPTER XVI</b>	
<b>Applications to Functions of Several Variables . . . . .</b>	<b>434</b>
§1. The Jacobian Matrix . . . . .	434
§2. Differentiability . . . . .	438
§3. The Chain Rule . . . . .	440
§4. Inverse Mappings . . . . .	443
§5. Implicit Functions . . . . .	446
§6. The Hessian . . . . .	450
<b>CHAPTER XVII</b>	
<b>The Change of Variables Formula . . . . .</b>	<b>453</b>
§1. Determinants as Area and Volume . . . . .	453
§2. Dilations . . . . .	463
§3. Change of Variables Formula in Two Dimensions . . . . .	469
§4. Application of Green's Formula to the Change of Variables Formula . . . . .	474
§5. Change of Variables Formula in Three Dimensions. . . . .	478
§6. Vector Fields on the Sphere . . . . .	483
<b>APPENDIX</b>	
<b>Fourier Series . . . . .</b>	<b>487</b>
§1. General Scalar Products . . . . .	487
§2. Computation of Fourier Series . . . . .	494
<b>Answers . . . . .</b>	<b>A1</b>
<b>Index . . . . .</b>	<b>I1</b>