## James O. Berger

## Statistical Decision Theory and Bayesian Analysis

Second Edition

With 23 Illustrations



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