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Uniqueness and Non-Uniqueness in the Cauchy Problem

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To Yasmine, Estelle and Stephane

INTRODUCTION

The question of uniqueness of the non-characteristic Cauchy problem, in the C^∞ framework, for more general operators than the hyperbolic ones, goes back to T. Carleman [11] who considered, in the two dimensional case, operators with real coefficients and simple complex characteristics. His proof, which was based on weighted estimates, turned out to be quite powerful since this method (called "Carleman estimates method") with various modifications is still today the only tool used to prove uniqueness theorems.

The first general results of the theory were given in 1957 by A. P. Calderon [8] as an interesting application of singular integral operators (or pseudo-differential operators). After that paper, several works were done (L. Hörmander [17], [18], S. Mizohata [32], R. Pederson[38], M. Protter [46]) giving new sufficient conditions for uniqueness and mainly devoted to elliptic operators.

Some time later, important progress was made by P. Cohen [12] and A. Pliš [40] to [45]. They gave several counterexamples which showed in particular that uniqueness does not hold for arbitrary elliptic operators, but depends of the multiplicities of the complex characteristics ; it depends also on all terms of the operator and on regularity of the coefficients. These authors gave, moreover, a way to construct non-trivial solutions, but this method was applied to particular examples or non invariant situations.

Then, L. Hörmander in his book, [19] chap. VIII, introduced the very important notion of strongly pseudo-convex hypersurfaces and proved uniqueness results for principally normal and elliptic operators.

In 1973 K. Watanabe [55] extended Calderon's theorem to certain operators with triple characteristics (the general case appeared later, in [57]).

L. Hörmander [20], trying to unify the counterexamples discussed above, gave in 1975 a more systematic way to construct non trivial solutions based on a very delicate variant of the geometrical optics method. However the results given there concerned perturbations of constant coefficient operators.

Since this time, using that method and the "Carleman estimates" it has been possible to go further. Indeed, the works of S. Alinhac-C. Zuily [5], R. Lascar-C. Zuily [27], S. Alinhac [1], [2], L. Hörmander [19] (Chap VIII) give almost necessary and sufficient conditions for uniqueness based on geometrical properties. Let us also mention the geometrical non-uniqueness theorem of S. Alinhac-M.S. Baouendi [4].

The aim of these notes is to present some aspects of the theory and to describe the tools used, mainly the non-uniqueness constructions. The plan is the following.

In the first Chapter, first order differential operators in $\mathbb{R}^n$ are considered. Using the works of M. Strauss-F. Treves [51] (with a simple proof) and S. Alinhac [1], an almost necessary and sufficient condition for uniqueness (for every zero order perturbation) is proved when $n \geqslant 3$. The two dimensional case is also discussed with positive results and counterexamples.

Chapter Two is devoted to the works initiated by Calderon's theorem. It begins by an improvement of his original work in the case of smooth characteristics. A case of non-smooth roots is also discussed (C. Zuily [65]). Then the result of K. Watanabe is given and a counterxample of A. Pliš [4] is discussed in detail. This chapter ends with some results concerning operators with characteristics of arbitrary high multiplicity. (K. Watanabe-C. Zuily [59], C. Zuily [64] etc...).

The third chapter begins by the operators with real principal part and Hörmander's uniqueness theorem ([19]) for strongly pseudo-convex hypersurfaces is proved. However the proof given here, due to N. Lerner [29], is a little more transparent, thanks to the use of the Weyl calculus for (pseudo) differential operators and an appropriate choice of the weight functions in the Carleman estimate. A non-uniqueness result (Alinhac [1]) is proved when the strong pseudo-convexity is violated (in a strong sense). Then we deal with a class of quasi-homogeneous second order operators and a geometrical (almost) characterization is given in terms of an appropriate notion of pseudo-convexity (R. Lascar-C. Zuily [27]). The third section of this chapter is devoted to the simplest case of operators with double real characteristics and a quite complete (and surprising) answer is given. (S. Alinhac-C. Zuily [5]). In section four we deal with elliptic operators; Hörmander's uniqueness theorem is proved by the method of N. Lerner [29] and a non-uniqueness result of S. Alinhac [2] is then stated. We end this chapter by several general non-uniqueness results stated without proof. ([1],[4],[20]).

In these notes we have considered only classical solutions, but several results could be extended to arbitrary distribution solutions (see for instance F. Cardoso - J. Hounie [10]). On the other hand we have not tried to give the minimal smoothness assumptions on the coefficients and these could be weakened. Moreover some material is missing : for instance uniqueness for constant coefficient operators (P.M. Goorjan [16], K. Watanabe [55] etc...) or non-uniqueness results in the case of non-smooth coefficients (in

Hölder classes for instance). For these kind of results we refer to P. Cohen [12], A. Pliš [44], L. Hörmander [20]. Let us remark that the material given here can be used in the characteristic Cauchy problem.

The bibliography at the end has been made as complete as possible, as far as the title of this book in its (more or less) strict sense is concerned.

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Orsay, October 1982.

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