

*Nonlinear Functional Analysis  
and its Applications*

*II/B: Nonlinear Monotone Operators*



*David Hilbert (1862–1943)*

Eberhard Zeidler

# Nonlinear Functional Analysis and its Applications

II/B: Nonlinear Monotone Operators

Translated by the Author and by Leo F. Boron†

With 74 Illustrations



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*To the memory of my parents*

# Preface to Part II/B

The present book is part of a comprehensive exposition of the main principles of nonlinear functional analysis and its numerous applications to the natural sciences and mathematical economics. The presentation is self-contained and accessible to a broader audience of mathematicians, natural scientists, and engineers. The material is organized as follows:

Part I: Fixed-point theorems.

Part II: Monotone operators.

Part III: Variational methods and optimization.

Parts IV/V: Applications to mathematical physics.

Here, Part II is divided into two subvolumes:

Part II/A: Linear monotone operators.

Part II/B: Nonlinear monotone operators.

These two subvolumes form a *unit* equipped with a uniform pagination. The contents of Parts II/A and II/B and the basic strategies of our presentation have been discussed in detail in the Preface to Part II/A. The present volume contains the complete index material for Parts II/A and II/B.

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