

Stephen Wiggins

# Introduction to Applied Nonlinear Dynamical Systems and Chaos

Second Edition

With 250 Figures



Springer

Stephen Wiggins  
School of Mathematics  
University of Bristol  
Clifton, Bristol BS8 1TW  
UK  
S.Wiggins@bristol.ac.uk

*Series Editors*

J.E. Marsden  
Control and Dynamical Systems, 107-81  
California Institute of Technology  
Pasadena, CA 91125  
USA  
marsden@cds.caltech.edu

L. Sirovich  
Division of Applied Mathematics  
Brown University  
Providence, RI 02912  
USA  
chico@camelot.mssm.edu

S.S. Antman  
Department of Mathematics  
*and*  
Institute for Physical Science  
and Technology  
University of Maryland  
College Park, MD 20742-4015  
USA  
ssa@math.umd.edu

Mathematics Subject Classification (2000): 58Fxx, 34Cxx, 70Kxx

Library of Congress Cataloging-in-Publication Data  
Wiggins, Stephen.

Introduction to applied nonlinear dynamical systems and chaos / Stephen Wiggins. — 2nd ed.  
p. cm. — (Texts in applied mathematics ; 2)  
Includes bibliographical references and index.  
ISBN 0-387-00177-8 (alk. paper)  
1. Differentiable dynamical systems. 2. Nonlinear theories. 3. Chaotic behavior in  
systems. I. Title. II. Texts in applied mathematics ; 2.  
QA614.8.W544 2003  
003'.85—dc21

2002042742

ISBN 0-387-00177-8

Printed on acid-free paper.

© 2003, 1990 Springer-Verlag New York, Inc.

All rights reserved. This work may not be translated or copied in whole or in part without the written permission of the publisher (Springer-Verlag New York, Inc., 175 Fifth Avenue, New York, NY 10010, USA), except for brief excerpts in connection with reviews or scholarly analysis. Use in connection with any form of information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed is forbidden.

The use in this publication of trade names, trademarks, service marks, and similar terms, even if they are not identified as such, is not to be taken as an expression of opinion as to whether or not they are subject to proprietary rights.

Printed in the United States of America.

9 8 7 6 5 4 3 2 1

SPIN 10901182

www.springer-ny.com

Springer-Verlag New York Berlin Heidelberg  
A member of BertelsmannSpringer Science+Business Media GmbH

# Contents

<b>Series Preface</b>	<b>v</b>
<b>Preface to the Second Edition</b>	<b>vii</b>
<b>Introduction</b>	<b>1</b>
<b>1 Equilibrium Solutions, Stability, and Linearized Stability</b>	<b>5</b>
1.1 Equilibria of Vector Fields . . . . .	5
1.2 Stability of Trajectories . . . . .	7
1.2a Linearization . . . . .	10
1.3 Maps . . . . .	15
1.3a Definitions of Stability for Maps . . . . .	15
1.3b Stability of Fixed Points of Linear Maps . . . . .	15
1.3c Stability of Fixed Points of Maps via the Linear Approximation . . . . .	15
1.4 Some Terminology Associated with Fixed Points . . . . .	16
1.5 Application to the Unforced Duffing Oscillator . . . . .	16
1.6 Exercises . . . . .	16
<b>2 Liapunov Functions</b>	<b>20</b>
2.1 Exercises . . . . .	25
<b>3 Invariant Manifolds: Linear and Nonlinear Systems</b>	<b>28</b>
3.1 Stable, Unstable, and Center Subspaces of Linear, Autonomous Vector Fields . . . . .	29
3.1a Invariance of the Stable, Unstable, and Center Subspaces . . . . .	32
3.1b Some Examples . . . . .	33
3.2 Stable, Unstable, and Center Manifolds for Fixed Points of Nonlinear, Autonomous Vector Fields . . . .	37
3.2a Invariance of the Graph of a Function: Tangency of the Vector Field to the Graph . . . . .	39
3.3 Maps . . . . .	40
3.4 Some Examples . . . . .	41
3.5 Existence of Invariant Manifolds: The Main Methods of Proof, and How They Work . . . . .	43

3.5a	Application of These Two Methods to a Concrete Example: Existence of the Unstable Manifold . . . .	45
3.6	Time-Dependent Hyperbolic Trajectories and their Stable and Unstable Manifolds . . . . .	52
3.6a	Hyperbolic Trajectories . . . . .	53
3.6b	Stable and Unstable Manifolds of Hyperbolic Trajectories . . . . .	56
3.7	Invariant Manifolds in a Broader Context . . . . .	59
3.8	Exercises . . . . .	62
<b>4</b>	<b>Periodic Orbits</b>	<b>71</b>
4.1	Nonexistence of Periodic Orbits for Two-Dimensional, Autonomous Vector Fields . . . . .	72
4.2	Further Remarks on Periodic Orbits . . . . .	74
4.3	Exercises . . . . .	76
<b>5</b>	<b>Vector Fields Possessing an Integral</b>	<b>77</b>
5.1	Vector Fields on Two-Manifolds Having an Integral . . . .	77
5.2	Two Degree-of-Freedom Hamiltonian Systems and Geometry . . . . .	82
5.2a	Dynamics on the Energy Surface . . . . .	83
5.2b	Dynamics on an Individual Torus . . . . .	85
5.3	Exercises . . . . .	85
<b>6</b>	<b>Index Theory</b>	<b>87</b>
6.1	Exercises . . . . .	89
<b>7</b>	<b>Some General Properties of Vector Fields: Existence, Uniqueness, Differentiability, and Flows</b>	<b>90</b>
7.1	Existence, Uniqueness, Differentiability with Respect to Initial Conditions . . . . .	90
7.2	Continuation of Solutions . . . . .	91
7.3	Differentiability with Respect to Parameters . . . . .	91
7.4	Autonomous Vector Fields . . . . .	92
7.5	Nonautonomous Vector Fields . . . . .	94
7.5a	The Skew-Product Flow Approach . . . . .	95
7.5b	The Cocycle Approach . . . . .	97
7.5c	Dynamics Generated by a Bi-Infinite Sequence of Maps . . . . .	97
7.6	Liouville's Theorem . . . . .	99
7.6a	Volume Preserving Vector Fields and the Poincaré Recurrence Theorem . . . . .	101
7.7	Exercises . . . . .	101
<b>8</b>	<b>Asymptotic Behavior</b>	<b>104</b>
8.1	The Asymptotic Behavior of Trajectories . . . . .	104

8.2	Attracting Sets, Attractors, and Basins of Attraction . . .	107
8.3	The LaSalle Invariance Principle . . . . .	110
8.4	Attraction in Nonautonomous Systems . . . . .	111
8.5	Exercises . . . . .	114
<b>9</b>	<b>The Poincaré-Bendixson Theorem</b>	<b>117</b>
9.1	Exercises . . . . .	121
<b>10</b>	<b>Poincaré Maps</b>	<b>122</b>
10.1	Case 1: Poincaré Map Near a Periodic Orbit . . . . .	123
10.2	Case 2: The Poincaré Map of a Time-Periodic Ordinary Differential Equation . . . . .	127
10.2a	Periodically Forced Linear Oscillators . . . . .	128
10.3	Case 3: The Poincaré Map Near a Homoclinic Orbit . . . . .	138
10.4	Case 4: Poincaré Map Associated with a Two Degree-of-Freedom Hamiltonian System . . . . .	144
10.4a	The Study of Coupled Oscillators via Circle Maps . . . . .	146
10.5	Exercises . . . . .	149
<b>11</b>	<b>Conjugacies of Maps, and Varying the Cross-Section</b>	<b>151</b>
11.1	Case 1: Poincaré Map Near a Periodic Orbit: Variation of the Cross-Section . . . . .	154
11.2	Case 2: The Poincaré Map of a Time-Periodic Ordinary Differential Equation: Variation of the Cross-Section . . . . .	155
<b>12</b>	<b>Structural Stability, Genericity, and Transversality</b>	<b>157</b>
12.1	Definitions of Structural Stability and Genericity . . . . .	161
12.2	Transversality . . . . .	165
12.3	Exercises . . . . .	167
<b>13</b>	<b>Lagrange's Equations</b>	<b>169</b>
13.1	Generalized Coordinates . . . . .	170
13.2	Derivation of Lagrange's Equations . . . . .	172
13.2a	The Kinetic Energy . . . . .	175
13.3	The Energy Integral . . . . .	176
13.4	Momentum Integrals . . . . .	177
13.5	Hamilton's Equations . . . . .	177
13.6	Cyclic Coordinates, Routh's Equations, and Reduction of the Number of Equations . . . . .	178
13.7	Variational Methods . . . . .	180
13.7a	The Principle of Least Action . . . . .	180
13.7b	The Action Principle in Phase Space . . . . .	182
13.7c	Transformations that Preserve the Form of Hamilton's Equations . . . . .	184
13.7d	Applications of Variational Methods . . . . .	186
13.8	The Hamilton-Jacobi Equation . . . . .	187

13.8a	Applications of the Hamilton-Jacobi Equation . . .	192
13.9	Exercises . . . . .	192
<b>14</b>	<b>Hamiltonian Vector Fields</b>	<b>197</b>
14.1	Symplectic Forms . . . . .	199
14.1a	The Relationship Between Hamilton's Equations and the Symplectic Form . . . . .	199
14.2	Poisson Brackets . . . . .	200
14.2a	Hamilton's Equations in Poisson Bracket Form . . .	201
14.3	Symplectic or Canonical Transformations . . . . .	202
14.3a	Eigenvalues of Symplectic Matrices . . . . .	203
14.3b	Infinitesimally Symplectic Transformations . . . . .	204
14.3c	The Eigenvalues of Infinitesimally Symplectic Matrices . . . . .	206
14.3d	The Flow Generated by Hamiltonian Vector Fields is a One-Parameter Family of Symplectic Transformations . . . . .	206
14.4	Transformation of Hamilton's Equations Under Symplectic Transformations . . . . .	208
14.4a	Hamilton's Equations in Complex Coordinates . . .	209
14.5	Completely Integrable Hamiltonian Systems . . . . .	210
14.6	Dynamics of Completely Integrable Hamiltonian Systems in Action-Angle Coordinates . . . . .	211
14.6a	Resonance and Nonresonance . . . . .	212
14.6b	Diophantine Frequencies . . . . .	217
14.6c	Geometry of the Resonances . . . . .	220
14.7	Perturbations of Completely Integrable Hamiltonian Systems in Action-Angle Coordinates . . . . .	221
14.8	Stability of Elliptic Equilibria . . . . .	222
14.9	Discrete-Time Hamiltonian Dynamical Systems: Iteration of Symplectic Maps . . . . .	223
14.9a	The KAM Theorem and Nekhoroshev's Theorem for Symplectic Maps . . . . .	223
14.10	Generic Properties of Hamiltonian Dynamical Systems . .	225
14.11	Exercises . . . . .	226
<b>15</b>	<b>Gradient Vector Fields</b>	<b>231</b>
15.1	Exercises . . . . .	232
<b>16</b>	<b>Reversible Dynamical Systems</b>	<b>234</b>
16.1	The Definition of Reversible Dynamical Systems . . . . .	234
16.2	Examples of Reversible Dynamical Systems . . . . .	235
16.3	Linearization of Reversible Dynamical Systems . . . . .	236
16.3a	Continuous Time . . . . .	236
16.3b	Discrete Time . . . . .	238

16.4	Additional Properties of Reversible Dynamical Systems . . . . .	239
16.5	Exercises . . . . .	240
<b>17</b>	<b>Asymptotically Autonomous Vector Fields</b>	<b>242</b>
17.1	Exercises . . . . .	244
<b>18</b>	<b>Center Manifolds</b>	<b>245</b>
18.1	Center Manifolds for Vector Fields . . . . .	246
18.2	Center Manifolds Depending on Parameters . . . . .	251
18.3	The Inclusion of Linearly Unstable Directions . . . . .	256
18.4	Center Manifolds for Maps . . . . .	257
18.5	Properties of Center Manifolds . . . . .	263
18.6	Final Remarks on Center Manifolds . . . . .	265
18.7	Exercises . . . . .	265
<b>19</b>	<b>Normal Forms</b>	<b>270</b>
19.1	Normal Forms for Vector Fields . . . . .	270
19.1a	Preliminary Preparation of the Equations . . . . .	270
19.1b	Simplification of the Second Order Terms . . . . .	272
19.1c	Simplification of the Third Order Terms . . . . .	274
19.1d	The Normal Form Theorem . . . . .	275
19.2	Normal Forms for Vector Fields with Parameters . . . . .	278
19.2a	Normal Form for The Poincaré-Andronov-Hopf Bifurcation . . . . .	279
19.3	Normal Forms for Maps . . . . .	284
19.3a	Normal Form for the Naimark-Sacker Torus Bifurcation . . . . .	285
19.4	Exercises . . . . .	288
19.5	The Elphick-Tirapegui-Brachet-Coullet-Iooss Normal Form . . . . .	290
19.5a	An Inner Product on $H_k$ . . . . .	291
19.5b	The Main Theorems . . . . .	292
19.5c	Symmetries of the Normal Form . . . . .	296
19.5d	Examples . . . . .	298
19.5e	The Normal Form of a Vector Field Depending on Parameters . . . . .	302
19.6	Exercises . . . . .	304
19.7	Lie Groups, Lie Group Actions, and Symmetries . . . . .	306
19.7a	Examples of Lie Groups . . . . .	308
19.7b	Examples of Lie Group Actions on Vector Spaces . . . . .	310
19.7c	Symmetric Dynamical Systems . . . . .	312
19.8	Exercises . . . . .	312
19.9	Normal Form Coefficients . . . . .	314
19.10	Hamiltonian Normal Forms . . . . .	316

19.10a General Theory . . . . . 316

19.10b Normal Forms Near Elliptic Fixed Points:  
     The Semisimple Case . . . . . 322

19.10c The Birkhoff and Gustavson Normal Forms . . . . . 333

19.10d The Lyapunov Subcenter Theorem  
     and Moser’s Theorem . . . . . 334

19.10e The KAM and Nekhoroshev Theorem’s Near an  
     Elliptic Equilibrium Point . . . . . 336

19.10f Hamiltonian Normal Forms and Symmetries . . . . . 338

19.10g Final Remarks . . . . . 342

19.11 Exercises . . . . . 342

19.12 Conjugacies and Equivalences of Vector Fields . . . . . 345

    19.12a An Application: The Hartman-Grobman  
         Theorem . . . . . 350

    19.12b An Application: Dynamics Near a Fixed  
         Point-Šořitařšvili’s Theorem . . . . . 353

19.13 Final Remarks on Normal Forms . . . . . 353

**20 Bifurcation of Fixed Points of Vector Fields 356**

20.1 A Zero Eigenvalue . . . . . 357

    20.1a Examples . . . . . 358

    20.1b What Is A “Bifurcation of a Fixed Point”? . . . . . 361

    20.1c The Saddle-Node Bifurcation . . . . . 363

    20.1d The Transcritical Bifurcation . . . . . 366

    20.1e The Pitchfork Bifurcation . . . . . 370

    20.1f Exercises . . . . . 373

20.2 A Pure Imaginary Pair of Eigenvalues:  
     The Poincare-Andronov-Hopf Bifurcation . . . . . 378

    20.2a Exercises . . . . . 386

20.3 Stability of Bifurcations Under Perturbations . . . . . 387

20.4 The Idea of the Codimension of a Bifurcation . . . . . 392

    20.4a The “Big Picture” for Bifurcation Theory . . . . . 393

    20.4b The Approach to Local Bifurcation Theory: Ideas  
         and Results from Singularity Theory . . . . . 397

    20.4c The Codimension of a Local Bifurcation . . . . . 402

    20.4d Construction of Versal Deformations . . . . . 406

    20.4e Exercises . . . . . 415

20.5 Versal Deformations of Families of Matrices . . . . . 417

    20.5a Versal Deformations of Real Matrices . . . . . 431

    20.5b Exercises . . . . . 435

20.6 The Double-Zero Eigenvalue: the Takens-Bogdanov  
     Bifurcation . . . . . 436

    20.6a Additional References and Applications for the  
         Takens-Bogdanov Bifurcation . . . . . 446

    20.6b Exercises . . . . . 446



20.7	A Zero and a Pure Imaginary Pair of Eigenvalues: the Hopf-Steady State Bifurcation . . . . .	449
20.7a	Additional References and Applications for the Hopf-Steady State Bifurcation . . . . .	477
20.7b	Exercises . . . . .	477
20.8	Versal Deformations of Linear Hamiltonian Systems . . . . .	482
20.8a	Williamson's Theorem . . . . .	482
20.8b	Versal Deformations of Jordan Blocks Corresponding to Repeated Eigenvalues . . . . .	485
20.8c	Versal Deformations of Quadratic Hamiltonians of Codimension $\leq 2$ . . . . .	488
20.8d	Versal Deformations of Linear, Reversible Dynamical Systems . . . . .	490
20.8e	Exercises . . . . .	491
20.9	Elementary Hamiltonian Bifurcations . . . . .	491
20.9a	One Degree-of-Freedom Systems . . . . .	491
20.9b	Exercises . . . . .	494
20.9c	Bifurcations Near Resonant Elliptic Equilibrium Points . . . . .	495
20.9d	Exercises . . . . .	497
<b>21</b>	<b>Bifurcations of Fixed Points of Maps</b>	<b>498</b>
21.1	An Eigenvalue of 1 . . . . .	499
21.1a	The Saddle-Node Bifurcation . . . . .	500
21.1b	The Transcritical Bifurcation . . . . .	504
21.1c	The Pitchfork Bifurcation . . . . .	508
21.2	An Eigenvalue of $-1$ : Period Doubling . . . . .	512
21.2a	Example . . . . .	513
21.2b	The Period-Doubling Bifurcation . . . . .	515
21.3	A Pair of Eigenvalues of Modulus 1: The Naimark-Sacker Bifurcation . . . . .	517
21.4	The Codimension of Local Bifurcations of Maps . . . . .	523
21.4a	One-Dimensional Maps . . . . .	524
21.4b	Two-Dimensional Maps . . . . .	524
21.5	Exercises . . . . .	526
21.6	Maps of the Circle . . . . .	530
21.6a	The Dynamics of a Special Class of Circle Maps-Arnold Tongues . . . . .	542
21.6b	Exercises . . . . .	550
<b>22</b>	<b>On the Interpretation and Application of Bifurcation Diagrams: A Word of Caution</b>	<b>552</b>

<b>23</b>	<b>The Smale Horseshoe</b>	<b>555</b>
23.1	Definition of the Smale Horseshoe Map . . . . .	555
23.2	Construction of the Invariant Set . . . . .	558
23.3	Symbolic Dynamics . . . . .	566
23.4	The Dynamics on the Invariant Set . . . . .	570
23.5	Chaos . . . . .	573
23.6	Final Remarks and Observations . . . . .	574
<b>24</b>	<b>Symbolic Dynamics</b>	<b>576</b>
24.1	The Structure of the Space of Symbol Sequences . . . . .	577
24.2	The Shift Map . . . . .	581
24.3	Exercises . . . . .	582
<b>25</b>	<b>The Conley–Moser Conditions, or “How to Prove That a Dynamical System is Chaotic”</b>	<b>585</b>
25.1	The Main Theorem . . . . .	585
25.2	Sector Bundles . . . . .	602
25.3	Exercises . . . . .	608
<b>26</b>	<b>Dynamics Near Homoclinic Points of Two-Dimensional Maps</b>	<b>612</b>
26.1	Heteroclinic Cycles . . . . .	631
26.2	Exercises . . . . .	632
<b>27</b>	<b>Orbits Homoclinic to Hyperbolic Fixed Points in Three-Dimensional Autonomous Vector Fields</b>	<b>636</b>
27.1	The Technique of Analysis . . . . .	637
27.2	Orbits Homoclinic to a Saddle-Point with Purely Real Eigenvalues . . . . .	640
27.2a	Two Orbits Homoclinic to a Fixed Point Having Real Eigenvalues . . . . .	651
27.2b	Observations and Additional References . . . . .	657
27.3	Orbits Homoclinic to a Saddle-Focus . . . . .	659
27.3a	The Bifurcation Analysis of Glendinning and Sparrow . . . . .	666
27.3b	Double-Pulse Homoclinic Orbits . . . . .	676
27.3c	Observations and General Remarks . . . . .	676
27.4	Exercises . . . . .	681
<b>28</b>	<b>Melnikov’s Method for Homoclinic Orbits in Two-Dimensional, Time-Periodic Vector Fields</b>	<b>687</b>
28.1	The General Theory . . . . .	687
28.2	Poincaré Maps and the Geometry of the Melnikov Function . . . . .	711
28.3	Some Properties of the Melnikov Function . . . . .	713

28.4	Homoclinic Bifurcations . . . . .	715
28.5	Application to the Damped, Forced Duffing Oscillator . . .	717
28.6	Exercises . . . . .	720
<b>29</b>	<b>Liapunov Exponents</b>	<b>726</b>
29.1	Liapunov Exponents of a Trajectory . . . . .	726
29.2	Examples . . . . .	730
29.3	Numerical Computation of Liapunov Exponents . . . . .	734
29.4	Exercises . . . . .	734
<b>30</b>	<b>Chaos and Strange Attractors</b>	<b>736</b>
30.1	Exercises . . . . .	745
<b>31</b>	<b>Hyperbolic Invariant Sets: A Chaotic Saddle</b>	<b>747</b>
31.1	Hyperbolicity of the Invariant Cantor Set $\Lambda$ Constructed in Chapter 25 . . . . .	747
31.1a	Stable and Unstable Manifolds of the Hyperbolic Invariant Set . . . . .	753
31.2	Hyperbolic Invariant Sets in $\mathbb{R}^n$ . . . . .	754
31.2a	Sector Bundles for Maps on $\mathbb{R}^n$ . . . . .	757
31.3	A Consequence of Hyperbolicity: The Shadowing Lemma . . . . .	758
31.3a	Applications of the Shadowing Lemma . . . . .	759
31.4	Exercises . . . . .	760
<b>32</b>	<b>Long Period Sinks in Dissipative Systems and Elliptic Islands in Conservative Systems</b>	<b>762</b>
32.1	Homoclinic Bifurcations . . . . .	762
32.2	Newhouse Sinks in Dissipative Systems . . . . .	774
32.3	Islands of Stability in Conservative Systems . . . . .	776
32.4	Exercises . . . . .	776
<b>33</b>	<b>Global Bifurcations Arising from Local Codimension—Two Bifurcations</b>	<b>777</b>
33.1	The Double-Zero Eigenvalue . . . . .	777
33.2	A Zero and a Pure Imaginary Pair of Eigenvalues . . . . .	782
33.3	Exercises . . . . .	790
<b>34</b>	<b>Glossary of Frequently Used Terms</b>	<b>793</b>
	<b>Bibliography</b>	<b>809</b>
	<b>Index</b>	<b>836</b>