Graduate Texts in Mathematics **61**

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George W. Whitehead

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In memory of
Norman Earl Steenrod (1910-1971)
and
John Henry Constantine Whitehead (1904–1960)

Preface

As the title suggests, this book is concerned with the elementary portion of the subject of homotopy theory. It is assumed that the reader is familiar with the fundamental group and with singular homology theory, including the Universal Coefficient and Künneth Theorems. Some acquaintance with manifolds and Poincaré duality is desirable, but not essential.

Anyone who has taught a course in algebraic topology is familiar with the fact that a formidable amount of technical machinery must be introduced and mastered before the simplest applications can be made. This phenomenon is also observable in the more advanced parts of the subject. I have attempted to short-circuit it by making maximal use of elementary methods. This approach entails a leisurely exposition in which brevity and perhaps elegance are sacrificed in favor of concreteness and ease of application. It is my hope that this approach will make homotopy theory accessible to workers in a wide range of other subjects—subjects in which its impact is beginning to be felt.

It is a consequence of this approach that the order of development is to a certain extent historical. Indeed, if the order in which the results presented here does not strictly correspond to that in which they were discovered, it nevertheless does correspond to an order in which they *might* have been discovered had those of us who were working in the area been a little more perspicacious.

Except for the fundamental group, the subject of homotopy theory had its inception in the work of L. E. J. Brouwer, who was the first to define the degree of a map and prove its homotopy invariance. This work is by now standard in any beginning treatment of homology theory. More subtle is the fact that, for self-maps of the n-sphere, the homotopy class of a map is characterized by its degree. An easy argument shows that it is sufficient to prove that any map of degree zero is homotopic to a constant map. The book begins, after a few pages of generalities, with Whitney's beautiful elementary proof of this fact. It may seem out of place to include a detailed proof so early in an introductory chapter. I have done so for two reasons: firstly, in order to have the result ready for use at the appropriate time, without breaking the line of thought; secondly, to emphasize the point (if emphasis be needed) that algebraic topology does not consist solely of the juggling of categories, functors and the like, but has some genuine geometric content.

Most of the results of elementary homotopy theory are valid in an arbitrary category of topological spaces. If one wishes to penetrate further into the subject, one encounters difficulties due to the failure of such properties as the exponential law, relating cartesian products and function spaces, to be universally valid. It was Steenrod who observed that, if one remains within the category of compactly generated spaces (this entails alteration of the standard topologies on products and function spaces), these difficulties evaporate. For this reason we have elected to work within this category from the beginning.

A critical role in homotopy theory is played by the homotopy extension property. Equally critical is the "dual", the homotopy lifting property. This notion is intimately connected with that of fibration. In the literature various notions of fibrations have been considered, but the work of Hurewicz has led to the "correct" notion: a fibre map is simply a continuous map which has the homotopy lifting property for arbitrary spaces.

The first chapter of the present work expounds the notions of the last three paragraphs. In Chapter II, relative CW-complexes are introduced. These were introduced, in their absolute form, by J. H. C. Whitehead, and it is clear that they supply the proper framework within which to study homotopy theory, particularly obstruction theory.

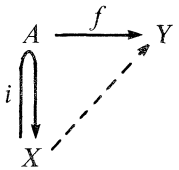
Chapter III is a "fun" chapter. After presenting evidence of the desirability of studying homotopy theory in a category of spaces with base points, the "dual" notions of H-spaces and H′-spaces are introduced. A space X is an H-space if and only if the set $[Y, X]$ of homotopy classes of maps of Y into X admits a law of composition which is natural with respect to maps of the domain; the definition of H′-space is strictly dual. H-spaces are characterized by the property that the folding map $X \vee X \to X$ can be extended over $X \times X$, while H′-spaces are characterized by the compressibility of the diagonal map $X \to (X \times X, X \vee X)$. The most important H′-spaces are the spheres, and the set $[S^n, Y] = \pi_n(Y)$ has a natural group structure, which is abelian if $n \geqq 2$.

Chapter IV takes up the systematic study of the homotopy groups $\pi_n(Y)$. Relative groups are introduced, and an exact sequence for the homotopy groups of a pair is established. Homotopy groups are seen to behave in many respects like homology groups; this resemblance is pointed up by the Hurewicz map, a homomorphism $\rho : \pi_n(X) \to H_n(X)$. The Hurewicz Theorem, which asserts that ρ is an isomorphism if X is $(n-1)$-connected, is proved. Homotopy groups behave particularly well for fibrations, and this fact facilitates the calculation of the first few homotopy groups of the classical groups.

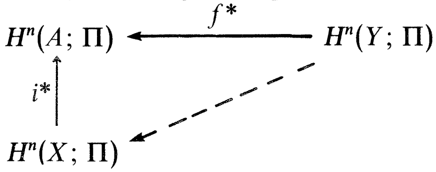
The fifth chapter is devoted to the homotopy properties of CW-complexes. The first half of the chapter is inspired by the work of J. H. C. Whitehead. The effect on the homotopy groups of the adjunction of a cell, or, more generally, the adjunction of a collection of cells of the same dimension, is considered. This allows one to construct a CW-complex with given homotopy groups. Moreover, if X is an arbitrary space, there is a CW-complex K and a map $f : K \to X$ which induces isomorphisms of the homotopy groups in all dimensions; i.e., f is a weak homotopy equivalence. Such a map is called a CW-approximation, and it induces isomorphisms of the homology groups as well. The device of CW-approximations allows one to replace the study of arbitrary spaces by that of CW-complexes.

The second part of Chapter V is concerned with obstruction theory. This powerful machinery, due to Eilenberg, is concerned with the *extension problem*: given a relative CW-complex (X, A) and a map $f : A \to Y$, does there exist an extension $g : X \to Y$ of f? This problem is attacked by a stepwise extension process: supposing that f has an extension g_n over the n-skeleton X_n of (X, A), one attempts to extend g_n over X_{n+1}. The attempt leads to an $(n+1)$-cochain c^{n+1} of (X, A) with coefficients in the group $\pi_n(Y)$. The fundamental property of the obstruction cochain c^{n+1} is that it is a cocycle whose cohomology class vanishes if and only if it is possible to alter g_n on the n-cells, without changing it on the $(n-1)$-skeleton, in such a way that the new map can be extended over X_{n+1}.

One can obtain further results by making simplifying assumptions on the spaces involved. One of the most important is the Hopf–Whitney Extension Theorem: if Y is $(n-1)$-connected and $\dim(X, A) \leq n+1$, then the extension problem



has a solution if and only if the algebraic problem



has a solution ($\Pi = \pi_n(Y)$).

Another important application occurs when Y is an Eilenberg–Mac Lane space $K(\Pi, n)$, i.e., $\pi_i(Y) = 0$ for all $i \neq n$. In this case, if X is an arbitrary CW-complex, then $[X, Y]$ is in one-to-one correspondence with the group $H^n(X; \Pi)$. In other words, the functor $H^n(\quad; \Pi)$ is *representable.*

The problem of finding a cross-section of a fibration $p: X \to B$ whose base space is a connected CW-complex can be attacked by similar methods. If $f: B_n \to X$ is a cross-section over B_n, the problem of extending f over an $(n+1)$-cell E_α gives rise to an element $c^{n+1}(e_\alpha) \in \pi_n(F_\alpha)$, where F_α is the fibre $p^{-1}(x_\alpha)$ over some point $b_\alpha \in \dot{E}_\alpha$. Now if b_0 and b_1 are points of B, the fibres $F_i = p^{-1}(b_i)$ have isomorphic homotopy groups; but the isomorphism is not unique, but depends on the choice of a homotopy class of paths in B from b_0 to b_1. Thus c^{n+1} is not a cochain in the usual sense. The machinery necessary to handle this more general situation is provided by Steenrod's theory of *homology with local coefficients.* A system G of local coefficients in a space B assigns to each $b \in B$ an abelian group $G(b)$ and to each homotopy class ξ of paths joining b_0 and b_1 an isomorphism $G(\xi): G(b_1) \to G(b_0)$. These are required to satisfy certain conditions which can be most concisely expressed by the statement that G is a functor from the fundamental groupoid of B to the category of abelian groups. To each space B and each system G of local coefficients in B there are then associated homology groups $H_n(B; G)$ and cohomology groups $H^n(B; G)$. These have properties very like those of ordinary homology and cohomology groups, to which they reduce when the coefficient system G is simple. These new homology groups are studied in Chapter VI. An important theorem of Eilenberg asserts that if B has a universal covering space $\tilde{B}$, the groups $H^n(B; G)$ are isomorphic with the *equivariant* homology and cohomology groups of $\tilde{B}$ with ordinary coefficients in $G_0 = G(b_0)$.

Having set up the machinery of cohomology with local coefficients the appropriate obstruction theory can be set up without difficulty; the obstructions c^{n+1} are cochains with coefficients in the system $\pi_n(\mathscr{F})$ of homotopy groups of the fibres. Results parallel to those of obstruction theory can then be proved. As an application, one may consider the universal bundle for the orthogonal group $\mathbf{O}_n$, whose base space is the Grassmannian of n-planes in $\mathbf{R}^\infty$. There are associated bundles whose fibres are the Stiefel manifolds $\mathbf{V}_{n,k}$, and the primary obstructions to the existence of cross-sections in these bundles are the Whitney characteristic classes.

If $F \to X \to B$ is a fibration, the relationships among the homotopy groups of the three spaces are expressed by an exact sequence. The behavior of the homology groups is much more complicated. In Chapter VII we study the behavior of the homology groups in certain cases which, while they are very special, nevertheless include a number of very important examples. In the first instance we assume that B is the suspension of a space

W and establish an exact sequence, the generalized Wang sequence, which expresses certain important relations among the homology groups of F, X and W. When X is contractible this allows us to calculate the homology groups of F by an induction on the dimension. When the coefficient group is a field, the result can be expressed by the statement that $H_*(F)$ is the tensor algebra over the graded vector space $H_*(W)$. The way the tensor algebra over a module M is built up out of M has its geometric analogue in the reduced product of James. Indeed, if X is a space with base point e, one forms the reduced product $J(X)$ by starting with the space of finite sequences of points of X and identifying two sequences if one can be obtained from the other by a finite number of insertions and deletions of the base point. The natural imbedding of X in ΩSX then extends to a map of $J(X)$ into ΩSX which is a weak homotopy equivalence. In particular, if X is a CW-complex, then $J(X)$ is a CW-approximation to ΩSX.

The case when B is a sphere is of special interest because the classical groups admit fibrations over spheres. The Wang sequence then permits us to calculate the cohomology rings (in fact, the cohomology Hopf algebras) for the most important coefficient domains.

Another case of special interest is that for which the fibre F is a sphere. When the fibration is orientable there is a *Thom isomorphism* $H^q(B) \approx H^{q+n+1}(\hat{X}, X)$, where $\hat{X}$ is the mapping cylinder of p. This leads to the Gysin sequence relating the cohomology groups of B and of X.

While the homology groups of F, X and B do not fit together to form an exact sequence, they do so in a certain range of dimensions. Specifically, if F is $(m-1)$-connected and B is $(n-1)$-connected, then $p_* : H_q(X, F) \to H_q(B)$ is an isomorphism for $q < m + n$ and an epimorphism for $q = m + n$. From this fact the desired exact sequence is constructed just as in the case of homotopy groups. This result is due to Serre; an important application is the Homotopy Excision Theorem of Blakers and Massey. To appreciate this result, let us observe that the homotopy groups do not have the Excision Property; i.e., if $(X; A, B)$ is a (nice) triad and $X = A \cup B$, the homomorphism

$$i_* : \pi_q(B, A \cap B) \to \pi_q(X, A)$$

induced by the inclusion map i is not, in general, an isomorphism. However, if $(A, A \cap B)$ is m-connected and $(B, A \cap B)$ in n-connected, then i_* is an isomorphism for $q < m + n$ and an epimorphism for $q = m + n$. The fact that this result can be deduced from the Serre sequence is due to Namioka. As a special case we have the Freudenthal Suspension Theorem: the homomorphism $E : \pi_q(S^n) \to \pi_{q+1}(S^{n+1})$ induced by the suspension operation is an isomorphism for $q < 2n - 1$ and an epimorphism for $q = 2n - 1$.

In Chapter V it was shown that the cohomology functor $H^n(\ \ ; \Pi)$ has a natural representation as $[\ \ , K(\Pi, n)]$. In a similar way, the natural transformations $H^n(\ \ ; \Pi) \to H^q(\ \ ; G)$ correspond to homotopy classes of maps between their representing spaces, i.e., to $[K(\Pi, n), K(G, q)] \approx H^q(K(\Pi, n); G)$. These *cohomology operations* are the object of study in Chapter VIII. Because the suspension $H^{r+1}(SX; A) \to H^r(X; A)$ is an isomorphism, each operation $\theta : H^n(\;\;; \Pi) \to H^q(\;\;; G)$ determines a new operation $H^{n-1}(\;\;; \Pi) \to H^{q-1}(\;\;; G)$, called the *suspension* of θ. By the remarks above the suspension can be thought of as a homomorphism $\sigma^* : H^q(\Pi, n; G) \to H^{q-1}(\Pi, n-1; G)$. Interpreting this homomorphism in the context of the path fibration

$$K(\Pi, n-1) = \Omega K(\Pi, n) \to \mathbf{P}K(\Pi, n) \to K(\Pi, n),$$

we deduce from the Serre exact sequence that σ^* is an isomorphism for $q < 2n$ and a monomorphism for $q = 2n$. Indeed, the homomorphisms σ^* can be imbedded in an exact sequence, valid in dimensions through $3n$. The remaining groups in the sequence are cohomology groups of $K(\Pi, n) \wedge K(\Pi, n)$, and interpretation of the remaining homomorphisms in the sequence yields concrete results on the kernel and cokernel of σ^*.

Examples of cohomology operations are the mod 2 Steenrod squares. They are a sequence of stable cohomology operations Sq^i $(i = 0, 1, \ldots)$. These are characterized by a few very simple properties. More sophisticated properties are due to Cartan and to Adem. The former are proved in detail; as for the latter, only a few instances are proved. With the aid of these results it follows that the Hopf fibrations $S^{2n-1} \to S^n$ and their iterated suspensions are essential; moreover, certain composites (for example $S^{n+2} \to S^{n+1} \to S^n$) of iterated Hopf maps are also.

Chapter VIII concludes with the calculation of the Steenrod operations in the cohomology of the classical groups (and the first exceptional group $\mathbf{G}_2$).

If X is an arbitrary (0-connected) space and N a positive integer, one can imbed X in a space X^N in such a way that (X^N, X) is an $(N+1)$-connected relative CW-complex and $\pi_q(X^N) = 0$ for all $q > N$. The pair (X^N, X) is unique up to homotopy type (rel. X); and the inclusion map $X \hookrightarrow X^{N+1}$ can be extended to a map of X^{N+1} into X^N, which is homotopically equivalent to a fibration having an Eilenberg–MacLane space $K(\pi_{N+1}(X), N+1)$ as fibre. The space X^{N+1} can be constructed from X^N with the aid of a certain cohomology class $k^{N+2} \in H^{N+2}(X^N; \pi_{N+1}(X))$. The system $\{X^N, k^{N+2}\}$ is called a Postnikov system for X, and the space X is determined up to weak homotopy type by its Postnikov system. The Postnikov system of X can be used to give an alternative treatment of obstruction theory for maps into X. These questions are treated in Chapter IX.

In Chapter X we return to the study of H-spaces. However, further conditions are imposed, in that the group axioms are assumed to hold up to homotopy. For such a space X the set $[Y, X]$ is a group for every Y. This group need not be abelian. However, under reasonable conditions it is nilpotent, and its nilpotency class is intimately related to the Lusternik–Schnirelmann category of Y. Of importance in studying these groups is the Samelson product. If $f : Y \to X$ and $g : Z \to X$ are maps, then the commutator map

$$(y, z) \to (f(y)g(z))(f(y)^{-1}g(z)^{-1})$$

of $Y \times Z$ into X is nullhomotopic on $Y \vee Z$ and therefore determines a well-defined homotopy class of maps of $Y \wedge Z$ into X. When Y and Z are spheres, so is $Y \wedge Z$, and we obtain a bilinear pairing $\pi_p(X) \otimes \pi_q(X) \to \pi_{p+q}(X)$. This pairing is commutative (up to sign) but is not associative. Instead one has a kind of Jacobi identity with signs.

Suppose, in particular, that X is the loop space of a space W. Then the isomorphisms $\pi_{r-1}(X) \approx \pi_r(W)$ convert the Samelson product in X to a pairing $\pi_p(W) \otimes \pi_q(W) \to \pi_{p+q-1}(W)$. This pairing is called the Whitehead product after its inventor, J. H. C. Whitehead, and the algebraic properties already deduced for the Samelson product correspond to like properties for that of Whitehead. Chapter X then concludes with a discussion of the relation between the Whitehead product and other operations in homotopy groups.

Chapter XI is devoted to homotopy operations. These are quite analogous to the cohomology operations discussed earlier. Universal examples for operations in several variables are provided by clusters of spheres

$$\Sigma = \mathbf{S}^{n_1} \vee \cdots \vee \mathbf{S}^{n_k}.$$

Indeed, each element $\alpha \in \pi_n(\Sigma)$ determines an operation $\theta_\alpha : \pi_{n_1} \times \cdots \times \pi_{n_k} \to \pi_n$ as follows. If $\alpha_i \in \pi_{n_i}(X)$ is represented by a map $f_i : \mathbf{S}^{n_i} \to X$ $(i = 1, \ldots, k)$, then the maps f_i together determine a map $f : \Sigma \to X$. We then define $\theta_\alpha(\alpha_1, \ldots, \alpha_k) = f_*(\alpha)$. And the map $\alpha \to \theta_\alpha$ is easily seen to be a one-to-one correspondence between $\pi_n(\Sigma)$ and the set of all operations having the same domain and range as θ_α.

Thus it is of importance to study the homotopy groups of a cluster of spheres. This was done by Hilton, who proved the relation

$$\pi_n(\Sigma) \approx \bigoplus_{r=1}^{\infty} \pi_n(\mathbf{S}^{n_r}),$$

where $\{n_r\}$ is a sequence of integers tending to ∞. The inclusion $\pi_n(\mathbf{S}^{n_r}) \to \pi_n(\Sigma)$ is given by $\beta \to \alpha_r \circ \beta$, where $\alpha_r \in \pi_{n_r}(\Sigma)$ is an iterated Whitehead product of the homotopy classes ι_j of the inclusion maps $\mathbf{S}^{n_j} \hookrightarrow \Sigma$ $(j = 1, \ldots, k)$. Hilton's theorem was generalized by Milnor in that the spheres $\mathbf{S}^{n_i}$ were replaced by arbitrary suspensions SX_i. Then Σ has to be replaced by SX, where $X = X_1 \vee \cdots \vee X_k$. The Hilton–Milnor Theorem then asserts that if the spaces X_i are connected CW-complexes, then $J(X)$ has the same homotopy type as the (weak) cartesian product

$$\prod_{r=1}^{\infty} J(X_r),$$

where X_r is an itered reduced join of copies of $X_1, \ldots, X_k$. The isomorphism in question is induced by a certain collection of iterated Samelson products.

One consequence of the Hilton Theorem is an analysis of the algebraic properties of the composition operation. The map $(\alpha, \beta) \to \beta \circ \alpha$ $(\alpha \in \pi_n(\mathbf{S}^r)$, $\beta \in \pi_r(X))$ is clearly additive in α, but it is not, in general, additive in β. The universal example here is $\beta = \iota_1 + \iota_2$, where ι_1 and ι_2 are the homotopy classes of the inclusions $\mathbf{S}^r \to \mathbf{S}^r \vee \mathbf{S}^r$. Application of the Hilton Theorem and naturality show that, if $\beta_1, \beta_2 \in \pi_r(X)$, then

$$(\beta_1 + \beta_2) \circ \alpha = \beta_1 \circ \alpha + \beta_2 \circ \alpha + \sum_{j=0}^{\infty} w_j(\beta_1, \beta_2) \circ h_j(\alpha),$$

where $w_j(\beta_1, \beta_2)$ is a certain iterated Whitehead product and $h_j : \pi_n(\mathbf{S}^r) \to \pi_n(\mathbf{S}^{n_j})$ is a homomorphism, the j^{th} *Hopf–Hilton homomorphism.*

The suspension operation induces a map of $[X, Y]$ into $[\mathbf{S}X, \mathbf{S}Y]$ for any spaces X, Y. We can iterate the procedure to obtain an infinite sequence

$$[X, Y] \to [\mathbf{S}X, \mathbf{S}Y] \to [\mathbf{S}^2X, \mathbf{S}^2Y] \to \cdots \to [\mathbf{S}^nX, \mathbf{S}^nY] \to \cdots$$

in which almost all of the sets involved are abelian groups and the maps homomorphisms. Thus we may form the direct limit

$$\{X, Y\} = \varinjlim_{n} [\mathbf{S}^nX, \mathbf{S}^nY];$$

it is an abelian group whose elements are called *S-maps* of X into Y. In particular, if $X = \mathbf{S}^n$, we obtain the n^{th} *stable homotopy group* $\sigma_n(Y) = \{\mathbf{S}^n, Y\}$.

We have seen that the homotopy and homology groups have many properties in common. The resemblance between stable homotopy groups and homology groups is even closer. Indeed, upon defining relative groups in the appropriate way, we see that they satisfy all the Eilenberg–Steenrod axioms for homology theory, *except for the Dimension Axiom.*

Examination of the Eilenberg–Steenrod axioms reveals that the first six axioms have a very general character, while the seventh, the Dimension Axiom, is very specific. In fact, it plays a normative role, singling out standard homology theory from the plethora of theories which satisfy the first six. That it is given equal status with the others is no doubt due to the fact that very few interesting examples of non-standard theories were known. But the developments of the last fifteen or so years has revealed the existence of many such theories: besides stable homotopy, one has the various K-theories and bordism theories.

Motivated by these considerations, we devote the remainder of Chapter XII to a discussion of homology theories without the dimension axiom. The necessity of introducing relative groups being something of a nuisance, we avoid it by reformulating the axioms in terms of a category of spaces with base point, rather than a category of pairs. The two approaches to homology theory are compared and shown to be completely equivalent.

The book might well end at this point. However, having eschewed the use of the heavy machinery of modern homotopy, I owe the reader a sample of things to come. Therefore a final chapter is devoted to the Leray–Serre spectral sequence and its generalization to non-standard homology theories. If $F \to X \to B$ is a fibration whose base is a CW-complex, the filtration of B by its skeleta induces one of X by their counterimages. Consideration of the homology sequences of these subspaces of X and their interrelations gives rise, following Massey, to an exact couple; the latter, in turn gives rise to a spectral sequence leading from the homology of the base with coefficients in the homology of the fibre to the homology of the total space. Some applications are given and the book ends by demonstrating the power of the machinery with some qualitative results on the homology of fibre spaces and on homotopy groups.

As I have stated, this book has been a mere introduction to the subject of homotopy theory. The rapid development of the subject in recent years has been made possible by more powerful and sophisticated algebraic techniques. I plan to devote a second volume to these developments.

The results presented here are the work of many hands. Much of this work is due to others. But mathematics is not done in a vacuum, and each of us must recognize in his own work the influence of his predecessors. In my own case, two names stand out above all the rest: Norman Steenrod and J. H. C. Whitehead. And I wish to acknowledge my indebtedness to these two giants of our subject by dedicating this book to their memory.

I also wish to express my indebtedness to my friends and colleagues Edgar H. Brown, Jr., Nathan Jacobson, John C. Moore, James R. Munkres, Franklin P. Peterson, Dieter Puppe, and John G. Ratcliffe, for reading portions of the manuscript and/or cogent suggestions which have helped me over many sticky points. Thanks are also due to my students in several courses based on portions of the text, particularly to Wensor Ling and Peter Welcher, who detected a formidable number of typographical errors and infelicities of style.

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June, 1978.

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