

Graduate Texts in Mathematics 83

Editorial Board

S. Axler F.W. Gehring K.A. Ribet

Springer-Science+Business Media, LLC

Graduate Texts in Mathematics

- 1 TAKEUTI/ZARING. Introduction to Axiomatic Set Theory. 2nd ed.
- 2 OXToby. Measure and Category. 2nd ed.
- 3 SCHAEFER. Topological Vector Spaces. 2nd ed.
- 4 HILTON/STAMMBACH. A Course in Homological Algebra. 2nd ed.
- 5 MAC LANE. Categories for the Working Mathematician. 2nd ed.
- 6 HUGHES/PIPER. Projective Planes.
- 7 SERRE. A Course in Arithmetic.
- 8 TAKEUTI/ZARING. Axiomatic Set Theory.
- 9 HUMPHREYS. Introduction to Lie Algebras and Representation Theory.
- 10 COHEN. A Course in Simple Homotopy Theory.
- 11 CONWAY. Functions of One Complex Variable I. 2nd ed.
- 12 BEALS. Advanced Mathematical Analysis.
- 13 ANDERSON/FULLER. Rings and Categories of Modules. 2nd ed.
- 14 GOLUBITSKY/GUILLEMIN. Stable Mappings and Their Singularities.
- 15 BERBERIAN. Lectures in Functional Analysis and Operator Theory.
- 16 WINTER. The Structure of Fields.
- 17 ROSENBLATT. Random Processes. 2nd ed.
- 18 HALMOS. Measure Theory.
- 19 HALMOS. A Hilbert Space Problem Book. 2nd ed.
- 20 HUSEMOLLER. Fibre Bundles. 3rd ed.
- 21 HUMPHREYS. Linear Algebraic Groups.
- 22 BARNES/MACK. An Algebraic Introduction to Mathematical Logic.
- 23 GREUB. Linear Algebra. 4th ed.
- 24 HOLMES. Geometric Functional Analysis and Its Applications.
- 25 HEWITT/STROMBERG. Real and Abstract Analysis.
- 26 MANES. Algebraic Theories.
- 27 KELLEY. General Topology.
- 28 ZARISKI/SAMUEL. Commutative Algebra. Vol. I.
- 29 ZARISKI/SAMUEL. Commutative Algebra. Vol. II.
- 30 JACOBSON. Lectures in Abstract Algebra I. Basic Concepts.
- 31 JACOBSON. Lectures in Abstract Algebra II. Linear Algebra.
- 32 JACOBSON. Lectures in Abstract Algebra III. Theory of Fields and Galois Theory.
- 33 HIRSCH. Differential Topology.
- 34 SPITZER. Principles of Random Walk. 2nd ed.
- 35 ALEXANDER/WERMER. Several Complex Variables and Banach Algebras. 3rd ed.
- 36 KELLEY/NAMIOKA et al. Linear Topological Spaces.
- 37 MONK. Mathematical Logic.
- 38 GRAUERT/FRITZSCHE. Several Complex Variables.
- 39 ARVESON. An Invitation to C^* -Algebras.
- 40 KEMENY/SNELL/KNAPP. Denumerable Markov Chains. 2nd ed.
- 41 APOSTOL. Modular Functions and Dirichlet Series in Number Theory. 2nd ed.
- 42 SERRE. Linear Representations of Finite Groups.
- 43 GILLMAN/JERISON. Rings of Continuous Functions.
- 44 KENDIG. Elementary Algebraic Geometry.
- 45 LOEVE. Probability Theory I. 4th ed.
- 46 LOEVE. Probability Theory II. 4th ed.
- 47 MOISE. Geometric Topology in Dimensions 2 and 3.
- 48 SACHS/WU. General Relativity for Mathematicians.
- 49 GRUENBERG/WEIR. Linear Geometry. 2nd ed.
- 50 EDWARDS. Fermat's Last Theorem.
- 51 KLINGENBERG. A Course in Differential Geometry.
- 52 HARTSHORNE. Algebraic Geometry.
- 53 MANIN. A Course in Mathematical Logic.
- 54 GRAVER/WATKINS. Combinatorics with Emphasis on the Theory of Graphs.
- 55 BROWN/PEARCY. Introduction to Operator Theory I: Elements of Functional Analysis.
- 56 MASSEY. Algebraic Topology: An Introduction.
- 57 CROWELL/FOX. Introduction to Knot Theory.
- 58 KOBLITZ. p -adic Numbers, p -adic Analysis, and Zeta-Functions. 2nd ed.
- 59 LANG. Cyclotomic Fields.
- 60 ARNOLD. Mathematical Methods in Classical Mechanics. 2nd ed.
- 61 WHITEHEAD. Elements of Homotopy

(continued after index)

Lawrence C. Washington

Introduction to Cyclotomic Fields

Second Edition



Springer

Lawrence C. Washington
Mathematics Department
University of Maryland
College Park, MD 20742
USA

Editorial Board

S. Axler
Mathematics Department
San Francisco State
University
San Francisco, CA 94132
USA

F.W. Gehring
Mathematics Department
East Hall
University of Michigan
Ann Arbor, MI 48109
USA

K.A. Ribet
Mathematics Department
University of California
at Berkeley
Berkeley, CA 94720-3840
USA

Mathematics Subject Classification (1991): 11Rxx, 11-01

With nine illustrations.

Library of Congress Cataloging-in-Publication Data
Washington, Lawrence C.

Introduction to cyclotomic fields / Lawrence C. Washington. — 2nd ed.

p. cm. — (Graduate texts in mathematics ; 83)

Includes bibliographical references and index.

ISBN 978-1-4612-7346-2 ISBN 978-1-4612-1934-7 (eBook)

DOI 10.1007/978-1-4612-1934-7

1. Algebraic fields. 2. Cyclotomy. I. Title. II. Series.

QA247.W35 1996

512'.74—dc20

96-13169

Printed on acid-free paper.

© 1997, 1982 Springer Science+Business Media New York

Originally published by Springer-Verlag New York in 1997, 1982

Softcover reprint of the hardcover 2nd edition 1997, 1982

All rights reserved. This work may not be translated or copied in whole or in part without the written permission of the publisher (Springer-Verlag New York, Inc., 175 Fifth Avenue, New York, NY 10010, USA), except for brief excerpts in connection with reviews or scholarly analysis. Use in connection with any form of information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed is forbidden.

The use of general descriptive names, trade names, trademarks, etc., in this publication, even if the former are not especially identified, is not to be taken as a sign that such names, as understood by the Trade Marks and Merchandise Marks Act, may accordingly be used freely by anyone.

Production managed by Robert Wexler; manufacturing supervised by Jacqui Ashri.

Typeset by Asco Trade Typesetting Ltd., Hong Kong.

9 8 7 6 5 4 3 2

ISBN 978-1-4612-7346-2

SPIN 10762507

To My Parents

Preface to the Second Edition

Since the publication of the first edition, several remarkable developments have taken place. The work of Thaine, Kolyvagin, and Rubin has produced fairly elementary proofs of Ribet's converse of Herbrand's theorem and of the Main Conjecture. The original proofs of both of these results used delicate techniques from algebraic geometry and were inaccessible to many readers. Also, Sinnott discovered a beautiful proof of the vanishing of Iwasawa's μ -invariant that is much simpler than the one given in Chapter 7. Finally, Fermat's Last Theorem was proved by Wiles, using work of Frey, Ribet, Serre, Mazur, Langlands–Tunnell, Taylor–Wiles, and others. Although the proof, which is based on modular forms and elliptic curves, is much different from the cyclotomic approaches described in this book, several of the ingredients were inspired by ideas from cyclotomic fields and Iwasawa theory.

The present edition includes two new chapters covering some of these developments. Chapter 15 treats the work of Thaine, Kolyvagin, and Rubin, culminating in a proof of the Main Conjecture for the p th cyclotomic field. Chapter 16 includes Sinnott's proof that $\mu = 0$ and his elementary proof of the corresponding result on the ℓ -part of the class number in a \mathbb{Z}_p -extension. Since the application of Jacobi sums to primality testing was too beautiful to omit, I have also included it in this chapter.

The first 14 chapters have been left essentially unchanged, except for corrections and updates. The proof of Fermat's Last Theorem, which is far beyond the scope of the present book, makes certain results of these chapters obsolete. However, I decided to let them remain, for they are interesting not only from an historical viewpoint but also as applications of various techniques. Moreover, some of the results of Chapter 9 apply to Vandiver's conjecture, one of the major unresolved questions in the field. For aesthetic reasons, it might have been appropriate to put the new Chapter 15 immedi-

ately after Chapter 13. However, I opted for the more practical route of placing it after the Kronecker–Weber theorem, thus ensuring that all numbering from the first edition is compatible with the second.

Other changes from the first edition include updating the bibliography and the addition of a table of class numbers of real cyclotomic fields due to Schoof.

Many people have sent me detailed lists of corrections and suggestions or have contributed in other ways to this edition. In particular, I would like to thank Brian Conrad, Keith Conrad, Li Guo, Mikihiro Hirabayashi, Jim Kraft, Tauno Metsänkylä, Ken Ribet, Yuan-Yuan Shen, Peter Stevenhagen, Patrick Washington, and Susan Zengerle.

Lawrence C. Washington

Preface to the First Edition

This book grew out of lectures given at the University of Maryland in 1979/1980. The purpose was to give a treatment of p -adic L -functions and cyclotomic fields, including Iwasawa's theory of \mathbb{Z}_p -extensions, which was accessible to mathematicians of varying backgrounds.

The reader is assumed to have had at least one semester of algebraic number theory (though one of my students took such a course concurrently). In particular, the following terms should be familiar: Dedekind domain, class number, discriminant, units, ramification, local field. Occasionally one needs the fact that ramification can be computed locally. However, one who has a good background in algebra should be able to survive by talking to the local algebraic number theorist. I have not assumed class field theory; the basic facts are summarized in an appendix. For most of the book, one only needs the fact that the Galois group of the maximal unramified abelian extension is isomorphic to the ideal class group, and variants of this statement.

The chapters are intended to be read consecutively, but it should be possible to vary the order considerably. The first four chapters are basic. After that, the reader willing to believe occasional facts could probably read the remaining chapters randomly. For example, the reader might skip directly to Chapter 13 to learn about \mathbb{Z}_p -extensions. The last chapter, on the Kronecker–Weber theorem, can be read after Chapter 2.

The notations used in the book are fairly standard; \mathbb{Z} , \mathbb{Q} , \mathbb{Z}_p , and \mathbb{Q}_p denote the integers, the rationals, the p -adic integers, and the p -adic rationals, respectively. If A is a ring (commutative with identity), then A^\times denotes its group of units. At Serge Lang's urging I have let the first Bernoulli number be $B_1 = -\frac{1}{2}$ rather than $+\frac{1}{2}$. This disagrees with Iwasawa [23] and several of my papers, but conforms to what is becoming standard usage.

Throughout the preparation of this book I have found Serge Lang's two volumes on cyclotomic fields very helpful. The reader is urged to look at them for different viewpoints on several of the topics discussed in the present volume and for a different selection of topics. The second half of his second volume gives a nice self-contained (independent of the remaining one and a half volumes) proof of the Gross–Koblitz relation between Gauss sums and the p -adic gamma function, and the related formula of Ferrero and Greenberg for the derivative of the p -adic L -function at 0, neither of which I have included here. I have also omitted a discussion of explicit reciprocity laws. For these the reader can consult Lang [4], Hasse [2], Henniart, Ireland–Rosen, Tate [3], or Wiles [1].

Perhaps it is worthwhile to give a very brief history of cyclotomic fields. The subject got its real start in the 1840s and 1850s with Kummer's work on Fermat's Last Theorem and reciprocity laws. The basic foundations laid by Kummer remained the main part of the theory for around a century. Then in 1958, Iwasawa introduced his theory of \mathbb{Z}_p -extensions, and a few years later Kubota and Leopoldt invented p -adic L -functions. In a major paper (Iwasawa [18]), Iwasawa interpreted these p -adic L -functions in terms of \mathbb{Z}_p -extensions. In 1979, Mazur and Wiles proved the Main Conjecture, showing that p -adic L -functions are essentially the characteristic power series of certain Galois actions arising in the theory of \mathbb{Z}_p -extensions.

What remains? Most of the universally accepted conjectures, in particular those derived from analogy with function fields, have been proved, at least for abelian extensions of \mathbb{Q} . Many of the conjectures that remain are probably better classified as "open questions," since the evidence for them is not very overwhelming, and there do not seem to be any compelling reasons to believe or not to believe them. The most notable are Vandiver's conjecture, the weaker statement that the p -Sylow subgroup of the ideal class group of the p th cyclotomic field is cyclic over the group ring of the Galois group, and the question of whether or not $\lambda = 0$ for totally real fields. In other words, we know a lot about imaginary things, but it is not clear what to expect in the real case. Whether or not there exists a fruitful theory remains to be seen.

Other possible directions for future developments could be a theory of $\hat{\mathbb{Z}}$ -extensions ($\hat{\mathbb{Z}} = \prod \mathbb{Z}_p$; some progress has recently been made by Friedman [1]), and the analogues of Iwasawa's theory in the elliptic case (Coates–Wiles [4]).

I would like to thank Gary Cornell for much help and many excellent suggestions during the writing of this book. I would also like to thank John Coates for many helpful conversations concerning Chapter 13. This chapter also profited greatly from the beautiful courses of my teacher, Kenkichi Iwasawa, at Princeton University. Finally, I would like to thank N.S.F. and the Sloan Foundation for their financial support and I.H.E.S. and the University of Maryland for their academic support during the writing of this book.

Lawrence C. Washington

Contents

Preface to the Second Edition	vii
Preface to the First Edition	ix
CHAPTER 1	
Fermat's Last Theorem	1
CHAPTER 2	
Basic Results	9
CHAPTER 3	
Dirichlet Characters	20
CHAPTER 4	
Dirichlet L -series and Class Number Formulas	30
CHAPTER 5	
p -adic L -functions and Bernoulli Numbers	47
5.1. p -adic functions	47
5.2. p -adic L -functions	55
5.3. Congruences	59
5.4. The value at $s = 1$	63
5.5. The p -adic regulator	70
5.6. Applications of the class number formula	77
	xi

CHAPTER 6	
Stickelberger's Theorem	87
6.1. Gauss sums	87
6.2. Stickelberger's theorem	93
6.3. Herbrand's theorem	100
6.4. The index of the Stickelberger ideal	102
6.5. Fermat's Last Theorem	107
CHAPTER 7	
Iwasawa's Construction of p-adic L-functions	113
7.1. Group rings and power series	113
7.2. p -adic L -functions	117
7.3. Applications	125
7.4. Function fields	128
7.5. $\mu = 0$	130
CHAPTER 8	
Cyclotomic Units	143
8.1. Cyclotomic units	143
8.2. Proof of the p -adic class number formula	151
8.3. Units of $\mathbb{Q}(\zeta_p)$ and Vandiver's conjecture	153
8.4. p -adic expansions	159
CHAPTER 9	
The Second Case of Fermat's Last Theorem	167
9.1. The basic argument	167
9.2. The theorems	173
CHAPTER 10	
Galois Groups Acting on Ideal Class Groups	185
10.1. Some theorems on class groups	185
10.2. Reflection theorems	188
10.3. Consequences of Vandiver's conjecture	196
CHAPTER 11	
Cyclotomic Fields of Class Number One	205
11.1. The estimate for even characters	206
11.2. The estimate for all characters	211

Contents	xiii
11.3. The estimate for h_m^-	217
11.4. Odlyzko's bounds on discriminants	221
11.5. Calculation of h_m^+	228
CHAPTER 12	
Measures and Distributions	232
12.1. Distributions	232
12.2. Measures	237
12.3. Universal distributions	252
CHAPTER 13	
Iwasawa's Theory of \mathbb{Z}_p -extensions	264
13.1. Basic facts	265
13.2. The structure of Λ -modules	269
13.3. Iwasawa's theorem	277
13.4. Consequences	285
13.5. The maximal abelian p -extension unramified outside p	292
13.6. The main conjecture	297
13.7. Logarithmic derivatives	301
13.8. Local units modulo cyclotomic units	312
CHAPTER 14	
The Kronecker–Weber Theorem	321
CHAPTER 15	
The Main Conjecture and Annihilation of Class Groups	332
15.1. Stickelberger's theorem	332
15.2. Thaine's theorem	334
15.3. The converse of Herbrand's theorem	341
15.4. The Main Conjecture	348
15.5. Adjoints	351
15.6. Technical results from Iwasawa theory	360
15.7. Proof of the Main Conjecture	369
CHAPTER 16	
Miscellany	373
16.1. Primality testing using Jacobi sums	373
16.2. Sinnott's proof that $\mu = 0$	380
16.3. The non- p -part of the class number in a \mathbb{Z}_p -extension	385

Appendix	391
1. Inverse limits	391
2. Infinite Galois theory and ramification theory	392
3. Class field theory	396
Tables	407
1. Bernoulli numbers	407
2. Irregular primes	410
3. Relative class numbers	412
4. Real class numbers	420
Bibliography	424
List of Symbols	483
Index	485