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From Number Theory to Physics

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With 93 Figures

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Preface

The present book contains fourteen expository contributions on various topics connected to Number Theory, or Arithmetics, and its relationships to Theoretical Physics. The first part is mathematically oriented; it deals mostly with elliptic curves, modular forms, zeta functions, Galois theory, Riemann surfaces, and p-adic analysis. The second part reports on matters with more direct physical interest, such as periodic and quasiperiodic lattices, or classical and quantum dynamical systems.

The contribution of each author represents a short self-contained course on a specific subject. With very few prerequisites, the reader is offered a didactic exposition, which follows the author's original viewpoints, and often incorporates the most recent developments. As we shall explain below, there are strong relationships between the different chapters, even though every single contribution can be read independently of the others.

This volume originates in a meeting entitled *Number Theory and Physics*, which took place at the Centre de Physique, Les Houches (Haute-Savoie, France), on March 7 – 16, 1989. The aim of this interdisciplinary meeting was to gather physicists and mathematicians, and to give to members of both communities the opportunity of exchanging ideas, and to benefit from each other's specific knowledge, in the area of Number Theory, and of its applications to the physical sciences. Physicists have been given, mostly through the program of lectures, an exposition of some of the basic methods and results of Number Theory which are the most actively used in their branch. Mathematicians have discovered in the seminars novel domains of Physics, where methods and results related to Arithmetics have been useful in the recent years.

The variety and abundance of the material presented during lectures and seminars led to the decision of editing two separate volumes, both published by Springer Verlag. The first book, entitled *Number Theory and Physics*, edited by J.M. Luck, P. Moussa, and M. Waldschmidt (Springer Proceedings in Physics, vol. **47**, 1990), contained the proceedings of the seminars, gathered into five parts: (I) Conformally Invariant Field Theories, Integrability, Quantum Groups; (II) Quasicrystals and Related Geometrical Structures; (III) Spectral Problems, Automata and Substitutions; (IV) Dynamical and Stochastic Systems; (V) Further Arithmetical Problems, and Their Relationship to Physics.

The present volume contains a completed and extended version of the lectures given at the meeting.

The central subject of Arithmetics is the study of the properties of rational integers. Deep results on this subject require the introduction of other sets. A first example is the ring of Gaussian integers

$$\mathbb{Z}[i] = \{a + ib; (a, b) \in \mathbb{Z}^2\},$$

related to the representation of integers as sums of two squares of integers. This ring has a rich arithmetic and analytic structure; it arises in this volume in many different guises: in chapter 1 in connection with quadratic forms, in chapters 2 and 3 as the group of periods of an elliptic function, in chapter 6 as a ring of integers of an algebraic number field, in chapter 7 it gives rise to a complex torus, in chapter 10 it is used as the main example of a lattice.

A second example is the field $\mathbb{C}$ of complex numbers which enables one to use methods from complex analysis. Analytic means have proved to be efficient in number theory; the most celebrated example is Riemann's zeta function, which provides information on the distribution of prime numbers.

There is a zeta function associated with the ring $\mathbb{Z}[i]$; it is constructed in chapter 1 by Cartier, in connection with quadratic forms, and also defined in chapter 6 by Stark, as the simplest case of the (Dedekind) zeta function of a number field. Other examples of Dirichlet series show up in this book: in chapter 1, Hurwitz zeta functions, in chapter 3, the Hasse-Weil zeta function of an elliptic curve, in chapter 4, the Hecke L-series attached to a modular form, in chapter 6, the Artin L-functions attached to a character; there are even p-adic L-functions in chapter 9. The mode-locking problem in chapter 13 involves another type of zeta function, which in some cases reduces to a ratio of two Riemann zeta functions.

Lattices, tori, and theta functions are also met in several chapters. The simplest lattice is $\mathbb{Z}$ in $\mathbb{R}$. The quotient is the circle (one-dimensional torus), which is studied in chapter 14. Lattices are intimately connected with elliptic curves and Abelian varieties (chapters 1, 2, 3, 5); they play an important role in Minkowski's geometry of numbers (chapter 10) and in Dirichlet's unit theorem (chapter 6). They arise naturally from the study of periodic problems, but their role extends to the study of quasiperiodic phenomena, especially in quasicrystallography (chapter 11). They deserve a chapter for their own (chapter 10). Theta functions were used by Jacobi to study sums of four squares of rational integers. They can be found in chapters 1, 2, 3, 5 and 10.

Let us now give a brief description of the content of each chapter.

In chapter 1 Cartier investigates properties of the Riemann's zeta function, with emphasis on its functional equation, by means of Fourier transformation, Poisson summation formula and Mellin transform. He also decomposes the zeta function attached to $\mathbb{Z}[i]$ into a product of the Riemann zeta function and a Dirichlet L-series with a character. This chapter includes exercises, which refer to more advanced results.

The set $\mathbb{Z}[i]$ is the simplest example of a lattice in the complex plane. When L is a general lattice in $\mathbb{C}$, the quotient group $\mathbb{C}/L$ can be given the structure of an algebraic Abelian group, which means that it is an algebraic variety, and that the group law is defined by algebraic equations.

If L is a lattice in higher dimension (a discrete subgroup of $\mathbb{C}^n$) the quotient $\mathbb{C}^n/L$ is not always an algebraic variety. Riemann gave necessary and sufficient conditions for the existence of a projective embedding of this torus as an Abelian variety. In this case theta functions give a complex parametrization. The correspondence between Riemann surfaces (see chapter 7), algebraic curves, and Jacobian varieties is explained by Bost in chapter 2. He surveys various definitions of Riemann surfaces, characterizes those defined over the field of algebraic numbers, discusses the notion of divisors and holomorphic bundles, including a detailed proof of the Riemann-Roch theorem. Various constructions of the Jacobian are presented, leading to the general theory of Abelian varieties. This thorough presentation can be viewed as an introduction to arithmetic varieties and Diophantine geometry.

Coming back to the one-dimensional case, a possible definition of an elliptic curve (chapter 3) is the quotient of $\mathbb{C}$ by a lattice L. The case $L = \mathbb{Z}[i]$ is rather special: the elliptic curve has non trivial endomorphisms. It is called of complex multiplication (CM) type. We are not so far from down-to-earth arithmetic questions; Cohen mentions a connection between the curve $y^2 = x^3 - 36x$ (which is 'an equation' of our elliptic curve) and the congruent number problem of finding right angle triangles with rational sides and given area. One is interested in the rational (or integral) solutions of such an equation. One method is to compute the number of solutions 'modulo p' for all prime numbers p. The collection of this data is recorded in an analytic function, which is another type of Dirichlet series, namely the L-function of the elliptic curve. According to Birch and Swinnerton-Dyer, this function contains (at least conjecturally) a large amount of arithmetic information.

Once the situation for a single elliptic curve is understood, one may wonder what happens if the lattice is varied. One thus comes across modular problems. A change of basis of a lattice involves an element of the modular group $\mathrm{SL}_2(\mathbb{Z})$, acting on the upper half plane. Once more analytic methods are relevant: one introduces holomorphic forms in the upper half plane, which satisfy a functional equation relating $f(\tau)$ to $f\big((a\tau + b)/(c\tau + d)\big)$. The modular invariant $j(\tau)$, the discriminant function Δ, Eisenstein series satisfy such a property. Taking sublattices induces transformations on these modular forms which are called Hecke operators. These operators act linearly on a vector space of modular forms; they have eigenvectors, and the collection of eigenvalues is included in a Dirichlet series, which is Hecke's L-series. An interesting special case is connected with the Δ function: the coefficient of q^n in the Fourier expansion is Ramanujan's τ function. In chapter 4 Zagier completes this introduction to modular forms by explaining the Eichler-Selberg trace formula which relates the trace of Hecke operators with the Kronecker-Hurwitz class number (which counts equivalence classes of binary quadratic forms with given discriminant).

In chapter 5 Gergondey also considers families of elliptic curves. He starts with the function

$$\vartheta_3(z \mid \tau) = \sum_{n\in\mathbb{Z}} e^{2i\pi\left(nz+(n^2/2)\tau\right)}$$

which is a solution of the heat equation. For fixed τ, this is an example of a theta function with respect to the lattice $\mathbb{Z}+\mathbb{Z}\tau$; quotients of such theta functions give a parametrization of points on an elliptic curve. For fixed z, the variable τ parametrizes lattices; but the so-called 'theta-constant' $\vartheta_3(\tau) = \vartheta_3(0 \mid \tau)$ is not a modular function: it is not invariant under isomorphisms of elliptic curves. The solution which is proposed is to change the notion of isomorphism by adding extra structures (it will be harder for two objects to be isomorphic: the situation will be rigidified). Moduli spaces thus obtained are nicer than without decoration.

Stark discusses in chapter 6 classical algebraic number theory. With almost no prerequisite one is taught almost the whole subject, including class field theory! He starts with Galois theory of algebraic extensions (with explicit examples: all subfields of $\mathbb{Q}(i, 2^{1/4})$ are displayed; another example involves a Jacobian of a curve of genus 2). He studies the ring of integers of an algebraic number field by means of divisor theory, avoiding abstract algebraic considerations. The Dedekind zeta function of a number field is introduced, with its functional equation, and its decomposition into a product of L-functions. Dirichlet class number formula proves once again the efficiency of analytic methods. The Cebotarev density theorem (which generalizes Dirichlet's theorem on primes in an arithmetic progression) is also included.

We mentioned that the quotient $\mathbb{C}/\mathbb{Z}[i]$ has the structure of an algebraic variety. This is a Riemann surface, and a meromorphic function on this surface is just an elliptic function. More generally, to a plane algebraic curve is associated such a surface, and coverings of curves give rise to extensions of function fields. Therefore Galois theory applies, as described by Reyssat in chapter 7. A useful way of computing the genus of a curve is the Riemann-Hurwitz formula. An application is mentioned to the inverse Galois problem: is it true that each finite group is the Galois group of an algebraic extension of $\mathbb{Q}$?

The quotient of the upper half plane by the modular group $\Gamma(7)$ is a curve of genus 3 with a group of 168 automorphisms; this curve is connected with a tessellation of the unit disc by hyperbolic triangles. By comparing Figure 12 of chapter 2, or Figure 14 of chapter 7, with the illustration of the front cover, the reader will realize easily that M.C. Escher's 'Angels and Demons' has been chosen for scientific reasons[1], and not because the meeting gathered Physicists and Mathematicians.

There is still a third type of Galois correspondence, in the theory of linear differential equations. This is explained in chapter 8 by Beukers (who could

[1] If one forgets about the difference between angels and demons, the group of hyperbolic isometries preserving the picture is generated by the (hyperbolic) mirror symmetries in the sides of a triangle with angles $\pi/2$, $\pi/4$, $\pi/6$. If one really wants to distinguish between angels and demons, one has to take a basic triangle which is twice as big, with angles $\pi/2$, $\pi/6$, $\pi/6$. Such groups are examples of Fuchsian groups associated with ternary quadratic forms, or quaternion algebras.

not attend the meeting). While the classical Galois theory deals with relations between the roots of an algebraic equations, differential Galois theory deals with algebraic relations between solutions of differential equations. Algebraic extensions of fields are now replaced by Picard-Vessiot extensions of differential fields. Kolchin's theorem provides a solution to the analytic problem which corresponds to solving algebraic equations with radicals. Another type of algebraic groups occurs here: the linear ones. Beukers also gives examples related to hypergeometric functions.

Analytic methods usually involve the field of complex numbers: $\mathbb{C}$ is the algebraic closure of $\mathbb{R}$, and $\mathbb{R}$ is the completion of $\mathbb{Q}$ for the usual absolute value. But $\mathbb{Q}$ has other absolute values, and ultrametric analysis is also a powerful tool. This is the object of chapter 9 by Christol. There is a large family of p-adic functions: exponential, logarithm, zeta and gamma functions, etc. Connections between the ring of adelic numbers $\widehat{\mathbb{Z}}$ and the Parisi matrices are pointed out.

Chapter 10, by Marjorie Senechal, deals with lattice geometry, a vast subject at the border between mathematics and physics, with applications ranging from integer quadratic forms to crystallography. The topics of Voronoï polytopes, root lattices and their Coxeter diagrams, and sphere packings, are covered in a more detailed fashion.

The next chapter is devoted to quasiperiodic lattices and tilings, which model the quasicrystalline phases, discovered experimentally in 1984. Katz uses the description of quasiperiodic sets of points as cuts of periodic objects in a higher-dimensional space. These objects are periodic arrays of 'atomic surfaces', which are placed at the vertices of a regular lattice. Several aspects of quasicrystallography are considered within this framework, including the Fourier transform and Patterson analysis, considerations about symmetry (point groups, self-similarity), and the possibility of growing a perfect quasiperiodic lattice from local 'matching rules'.

The last three chapters involve concepts and results related to the theory of dynamical systems, in a broad sense, namely, the study of temporal evolution, according to the laws of either classical or quantum mechanics.

In chapter 12, Bellissard presents an overview of the consequences of algebraic topology, and especially K-theory, on the spectra of Hamiltonian or evolution operators in quantum mechanics. The main topic is the gap labelling problem. Several applications are discussed, including the propagation of electrons on a lattice in a strong magnetic field, the excitation spectra of quasicrystals, and various one-dimensional spectral problems, in connection with sequences generated by automata or substitutions.

Cvitanović deals with circle maps in chapter 13. These provide examples of classical dynamical systems which are both simple enough to allow for a detailed and comprehensive study, and complex enough to exhibit the many features referred to as 'chaos'. The mathematical framework of this field involves approximation theory for irrational numbers (continued fraction expansions, Farey series).

The last chapter is devoted to yet a different aspect of dynamical systems, known as 'small divisor problems'. This name originates in the occurrence of small divisors in the calculations of the stability of a periodic orbit of a Hamiltonian dynamical system under small perturbations. Yoccoz reviews the progress made in the understanding of the behavior of periodic orbits throughout this century, starting with the pioneering works by Poincaré and Denjoy.

Let us finally emphasize that the title of this book reveals our conviction that number-theoretical concepts are becoming more and more fruitful in many areas of the natural sciences, as witnessed by the success of the meeting during which part of the material of this book has been presented.

Saclay and Paris, May 1992.

M. Waldschmidt, P. Moussa, J.M. Luck, C. Itzykson

We take advantage of this opportunity to thank again Prof. N. Boccara, the Director of the Centre de Physique, for having welcomed the meeting on the premises of the Les Houches School, with its unique atmosphere, in a charming mountainous setting, amongst ski slopes (see 'Quasicrystals: The View from Les Houches', by M. Senechal and J. Taylor, The Mathematical Intelligencer, vol. 12, pp. 54–64, 1990).

The Scientific Committee which organized *Number Theory and Physics* was composed of: J. Bellissard (Theoretical Physics, Toulouse), C. Godrèche (Solid State Physics, Saclay), C. Itzykson (Theoretical Physics, Saclay), J.M. Luck (Theoretical Physics, Saclay), M. Mendès France (Mathematics, Bordeaux), P. Moussa (Theoretical Physics, Saclay), E. Reyssat (Mathematics, Caen), and M. Waldschmidt (Mathematics, Paris). The organizers have been assisted by an International Advisory Committee, composed of Profs. M. Berry (Physics, Bristol, Great-Britain), P. Cvitanović (Physics, Copenhagen, Denmark), M. Dekking (Mathematics, Delft, The Netherlands), and G. Turchetti (Physics, Bologna, Italy).

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