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Triangular Products of Group Representations and Their Applications

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INTRODUCTION

The construction considered in these notes is based on a very simple idea. Let (A, G_1) and (B, G_2) be two group representations, for definiteness faithful and finite-dimensional, over an arbitrary field. We shall say that a faithful representation (V, G) is an extension of (A, G_1) by (B, G_2) if there is a G-submodule W of V such that the naturally arising representations (W, G) and $(V/W, G)$ are isomorphic, modulo their kernels, to (A, G_1) and (B, G_2) respectively.

Question. Among all the extensions of (A, G_1) by (B, G_2), does there exist such a "universal" extension which contains an isomorphic copy of any other one?

The answer is in the affirmative. Really, let $\dim A = m$ and $\dim B = n$, then the groups G_1 and G_2 may be considered as matrix groups of degrees m and n respectively. If (V, G) is an extension of (A, G_1) by (B, G_2) then, under certain choice of a basis in V, all elements of G are represented by $(m + n) \times (m + n)$ matrices of the form

$$\left(\begin{array}{c|c} g_1 & 0 \\ \hline h & g_2 \end{array}\right) \qquad (*)$$

where $g_i \in G_i$. It is now clear that the natural representation $(A \oplus B, G^*)$ where G^* is the group of all possible matrices of the form $(*)$ is the desired "universal" extension. Forestalling events, let us say that it is the representation $(A \oplus B, G^*)$ that is called the triangular product of the representations (A, G_1) and (B, G_2). It is the simplest example of this construction which, in general, relates to arbitrary representations over arbitrary commutative rings.

Despite its naturality and transparency, the construction of triangular product appeared in an explicit form only in 1971. It was introduced by Plotkin [50] with the purpose of investigating the semigroup of varieties of group representations. Very soon the construction turned out to have a number of interesting applications in other fields of algebra, and it was successfully used by many authors. Unfortunately, the corresponding results are rather disorderly scattered over the literature, some of them have been published in the editions which are not widely distributed (this also relates to the initial Plotkin's paper), but some of them have not yet been published at all. This explains why a new man wishing to acquire the subject encounters a number of difficulties, not only of mathematical nature.

The aim of the present notes is threefold. First, we try to present a detailed and self-contained account of what have been already done in the area. Second, we hope to persuade the reader that the technique of triangular products may be successfully applied to the investigation of quite concrete and long-familiar algebraic objects, such as augmentation ideals and dimension subgroups, triangular matrix groups and algebras, representations of lattices. Finally, we would like to point out some open questions which seem to be rather interesting.

Although the greater part of the material in these notes is not new, there is a number of places where the existing work has been simplified or generalized. Besides, there are several novel results - for instance, Theorems 9,8, 12.1 and their corollaries, Proposition 8.5.

The paper consists of two chapters. Chapter 1 deals with triangular products themselves, and its results show that this construction is of certain own interest. First, the triangular product is a functor from the category of group representations to itself which, under certain hypotheses, is left or right exact. Further, the triangular product can be defined as a universal object in a special category. It is also of interest that any two triangular decompositions of a given faithful representation have mutually conjugated refinements - the result which to some extent reminds of the classical Krull-Remak-Schmidt Theorem. Finally, the triangular product agrees very well with the multiplication of varieties of group representations: for instance, if ϱ_1 and ϱ_2 are arbitrary representations over a field, then

$$\operatorname{var}(\varrho_1 \triangledown \varrho_2) = \operatorname{var} \varrho_1 \cdot \operatorname{var} \varrho_2.$$

Chapter 2 is devoted to various applications of triangular products. Without going into details, let us note here some typical consequences which follow from the main results of this chapter.

1. The semigroup of T-ideals of an absolutely free associative algebra over a field is free.

2. If K is an integral domain, then the following identity forms a basis for the identities of the full triangular representation $(K^n, T_n(K))$:

$$(1 - [x_1, y_1]) \ldots (1 - [x_n, y_n]) \qquad \text{if } |K^*| = \infty,$$
$$(1 - [x_1, y_1]z_1^m) \ldots (1 - [x_n, y_n]z_n^m) \quad \text{if } |K^*| = m < \infty.$$

3. If K is an integral domain, a complete description of the identities of the full triangular group $T_n(K)$ is given; for example, if K is a field of characteristic

0, then these identities are generated by

$$[[x_1, y_1], [x_2, y_2], \dots, [x_n, y_n]].$$

4. Let K be a Dedekind domain, P its field of fractions and F an absolutely free group. If I and J are completely invariant ideals of the group algebra PF, then

$$IJ \cap KF = (I \cap KF)(J \cap KF).$$

5. For every integer $n \geqslant 0$ there exists a finite group G such that the augmentation terminal of the group ring $\mathbb{Z}G$ is not less than $\omega + n$ where ω is the first infinite ordinal.

6. For every group G and every field K

$$\delta_{\omega+1}(G, K) = \delta_{\omega+2}(G, K) = \dots = \delta_{\omega 2}(G, K);$$

here $\delta_\alpha(G, K)$ is the α-th dimension subgroup of G over K. The full description of these subgroups in group theoretic terms is given.

7. If G is a periodic group, then

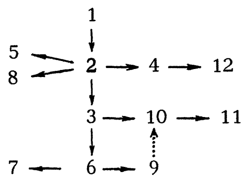
$$\delta_{\omega+1}(G, \mathbb{Z}) \subseteq \gamma_\omega(G)$$

where $\gamma_\omega(G)$ is the ω-th term of the lower central series of G.

8. If L is a finite distributive lattice, then over an arbitrary field there exists a representation (V, G) such that the lattice of all G-submodules of V is isomorphic to L.

The notes are completed with a brief appendix on triangular products of certain other objects closely related to group representations. It begins with triangular products of representations of associative and Lie algebras. The corresponding definitions, theorems and proofs are, as a rule, completely analogous to their "group" prototypes. In particular, this allows to obtain in a uniform manner certain results on groups, associative algebras and Lie algebras which were earlier proved separately in each concrete situation (note, for instance, the description of the identities of triangular matrix groups and algebras). In conclusion we consider triangular products of linear automata and discuss a little their applications to decomposition problems, which turned out to be rather encouraging.

One object of these notes is to present all of the theory in a form understandable for many people. So, only the standard algebra background (say, Lang's "Algebra") plus a little bit from the theory of varieties of algebraic structures is presupposed of the reader. Furthermore, we also try to avoid long chains of logical cosequences: except Sections 1 and 2, on which the whole paper is based, all the other sections can be read more or less independently according to the following scheme:



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