

# Contents

Preface	xi
Chapter 1. Forms and Galois Cohomology	1
§1 Group schemes and their cohomology	
1.1 Group objects in a category	1
1.2 Group schemes	3
1.3 Affine groups, Hopf algebras	4
1.4 Group schemes over a field, algebraic groups	7
1.5 Frobenius morphisms	7
1.6 Diagonal groups	10
1.7 Characters of group schemes	11
1.8 Bicharacters	13
1.9 Exactness of the functor $D$	14
1.10 Galois cohomology	16
1.11 Sheaves and cohomology in the étale topology	17
1.12 Cartier divisors and Weil divisors	19
§2 The Brauer group of a projective variety	20
2.1 The unramified Brauer group of a function field	20
2.2 The Kummer exact sequence	20
2.3 The Tate group, the Picard number, the Lefschetz number	21
§3 The theory of $k$ -forms	23
3.1 Forms and one-dimensional cohomology	23
3.2 Splitting fields of a $k$ -form	24
3.3 Forms of group schemes	25
3.4 Groups of multiplicative type	25
3.5 Principal homogeneous spaces	27
3.6 Projective groups and associated $k$ -forms	29
3.7 The Brauer group of a field	30
3.8 Chevalley groups	32
3.9 Semisimple groups	35
3.10 Inner and outer forms	36
3.11 Almost simple semisimple groups	37
3.12 The Weil restriction	37
Chapter 2. Birational Geometry of Algebraic Tori	41
§4 Birational invariants of linear algebraic groups	41
4.1 The variety of maximal tori of a reductive group	41
4.2 Structure of the generic torus of a semisimple group	42

4.3	The Picard group and the Brauer group of a linear algebraic group	44
4.4	Criteria for birational equivalence of algebraic varieties	46
4.5	Projective models of linear algebraic groups	47
4.6	Flasque resolutions of a module	49
4.7	Stable equivalence	51
4.8	Chevalley modules	52
4.9	Tori of small dimension	57
4.10	Tori with a biquadratic splitting field	58
4.11	The semigroup of stable equivalence	59
§5	Tori with a cyclic splitting field	60
5.1	“Dévissage” of a quasi-split torus	60
5.2	Invertibility of the Picard class	62
5.3	The Chistov multiplication	62
§6	Stable rationality of varieties	65
6.1	Stably rational tori as orbit varieties	65
6.2	Covariants of linear representations	67
6.3	Rationality of tori of type $pq$	69
6.4	Universal torsors	71
6.5	Counterexamples to Zariski’s conjecture	73
Chapter 3.	Invariants of Finite Transformation Groups	75
§7	Fields of invariants of finite transformation groups	75
7.1	Fields of invariants and their models	75
7.2	Invariants of finite abelian groups	76
7.3	The fields $(k, p^\alpha)$ , $p > 2$	78
7.4	The fields $(k, 2^\alpha)$	79
7.5	General case	79
7.6	Invariants of finite groups over an algebraically closed field	81
7.7	Invariants of finite linear groups	82
7.8	Invariants of finite groups acting on tori	85
7.9	Invariants of connected algebraic groups	87
§8	Invariant projective Demazure models	90
8.1	Cones and fans	90
8.2	Projective invariant fans	93
8.3	Birational invariants of tori without affect	97
8.4	The graded ring of a toric variety	99
Chapter 4.	Arithmetic of Linear Algebraic Groups	103
§9	Tori over a finite field	103
9.1	Number of rational points	103
9.2	Zeta function	104
§10	Tori over local fields	106
10.1	Tori over reals	106
10.2	Tori over a nonarchimedean field	107
10.3	Integer structures in linear algebraic groups	107
10.4	Canonical integer form of a quasisplit torus	109
10.5	Canonical form of a norm torus	111
§11	Tori over global fields	111

11.1	Adele groups	111
11.2	Canonical integer model of a torus over a number field	113
11.3	Cohomology of adèle groups	114
11.4	Descent of the ground field	118
11.5	Approximation problems	119
11.6	Arithmetical meaning of the birational invariant $H^1(k, p(T))$	120
§12	Arithmetic of semisimple groups	122
12.1	Cohomology of semisimple groups	122
12.2	Weak approximation	124
12.3	The group $H^1(k, \text{Pic } \overline{X})$	125
§13	Artin $L$ -functions	127
13.1	Partial Artin $L$ -functions	127
13.2	Theorems of Artin and Brauer	129
13.3	Global zeta function of a torus	131
Chapter 5.	Tamagawa Numbers	133
§14	Haar measure on adèle groups	133
14.1	Product of local measures	133
14.2	Computation of local volumes	134
14.3	Canonical convergence factors	136
14.4	The Tamagawa measure	137
14.5	Properties of Tamagawa numbers	142
14.6	Tamagawa numbers of algebraic tori	142
14.7	The group $\Phi$	147
14.8	Further development of the method	148
14.9	Chevalley group $\mathbb{Z}$ -schemes	148
14.10	Gindikin–Karpelevich integrals	149
14.11	Langlands' method of computing Tamagawa numbers	153
14.12	Elementary computations of volumes of some classical quotients	160
§15	The Minkowski–Siegel–Tamagawa formula	163
15.1	Infinite products	163
15.2	The weight of a genus of an odd positive lattice	165
15.3	The weight of a genus of an even positive unimodular lattice	169
15.4	Sums of squares	169
15.5	Sum of two squares	172
15.6	Sum of four squares	173
15.7	Sum of six squares	173
15.8	Sum of eight squares	174
15.9	Sum of three squares	174
15.10	Sum of five squares	175
15.11	Sum of seven squares	176
Chapter 6.	$R$ -equivalence in Algebraic Groups	177
§16	The group of $R$ -equivalence classes	177
16.1	First properties of $R$ -equivalence on varieties	177
16.2	Birational invariance of $R$ -equivalence in groups	179
§17	$R$ -equivalence on algebraic tori	180
17.1	Flasque resolution of a torus and $R$ -equivalence	180
17.2	Some special tori	182

17.3	The group $T(k(t))$	184
§18	The unimodular group of a simple algebra	185
18.1	Reduction to the anisotropic kernel	185
18.2	The Whitehead group of a simple algebra	185
18.3	Platonov's examples	187
18.4	The Whitehead group of an isotropic group	188
18.5	$R$ -equivalence over special fields	188
§19	Algebras with involutions and groups of adjoint type	190
19.1	Algebras with involutions	190
19.2	Indecomposable algebras with involutions	190
19.3	Automorphisms of indecomposable algebras with involutions	192
19.4	Forms of algebras with involutions	192
19.5	The covering of $G_0$	193
19.6	Merkurjev's theorems	194
Chapter 7.	Index Formulas in Arithmetic of Algebraic Tori	197
§20	Arithmetic of the projective group of a field	197
20.1	Ratio of class numbers	197
20.2	Index formulas for quadratic extensions	200
20.3	The Hasse relations for an imaginary extension	201
§21	Arithmetic of a norm hypersurface	202
	Bibliographical Remarks	207
	Bibliography	211