

Topics in Optimal Transportation

Cédric Villani

**Graduate Studies
in Mathematics**

Volume 58



**American Mathematical Society
Providence, Rhode Island**

Contents

Preface	ix
Notation	xiii
Introduction	1
1. Formulation of the optimal transportation problem	1
2. Basic questions	6
3. Overview of the course	10
Chapter 1. The Kantorovich Duality	17
1.1. General duality	17
1.2. Distance cost functions	34
1.3. Appendix: A duality argument in $C_b(X \times Y)$	39
1.4. Appendix: $\{0, 1\}$ -valued costs and Strassen's theorem	44
Chapter 2. Geometry of Optimal Transportation	47
2.1. A duality-based proof for the quadratic cost	48
2.2. The real line	73
2.3. Alternative arguments	78
2.4. Generalizations to other costs	85
2.5. More on c -concave functions	103
Chapter 3. Brenier's Polar Factorization Theorem	107
3.1. Rearrangements and polar factorization	107
3.2. Historical motivations: fluid mechanics	111
3.3. Proof of Brenier's polar factorization theorem	119

3.4. Related facts	122
Chapter 4. The Monge-Ampère Equation	125
4.1. Informal presentation	125
4.2. Regularity	131
4.3. Open problems	141
Chapter 5. Displacement Interpolation and Displacement Convexity	143
5.1. Displacement interpolation	143
5.2. Displacement convexity	150
5.3. Application: uniqueness of ground state	163
5.4. The Eulerian point of view	165
Chapter 6. Geometric and Gaussian Inequalities	183
6.1. Brunn-Minkowski and Prékopa-Leindler inequalities	184
6.2. The Alesker-Dar-Milman diffeomorphism	190
6.3. Gaussian inequalities	192
6.4. Sobolev inequalities	200
Chapter 7. The Metric Side of Optimal Transportation	205
7.1. Monge-Kantorovich distances	207
7.2. Topological properties	212
7.3. The real line	218
7.4. Behavior under rescaled convolution	220
7.5. An application to the Boltzmann equation	223
7.6. Linearization	233
Chapter 8. A Differential Point of View on Optimal Transportation	237
8.1. A differential formulation of optimal transportation	238
8.2. Differential calculus	250
8.3. Monge-Kantorovich induced dynamics	251
8.4. Time-discretization	256
8.5. Differentiability of the quadratic Wasserstein distance	262
8.6. Non-quadratic costs	266
Chapter 9. Entropy Production and Transportation Inequalities	267
9.1. More on optimal-transportation induced dissipative equations	268

9.2. Logarithmic Sobolev inequalities	279
9.3. Talagrand inequalities	291
9.4. HWI inequalities	297
9.5. Nonlinear generalizations: internal energy	301
9.6. Nonlinear generalizations: interaction energy	304
Chapter 10. Problems	307
List of Problems	308
Bibliography	349
Table of Short Statements	363
Index	367