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Ferdinand Verhulst

# Nonlinear Differential Equations and Dynamical Systems

With 107 Figures

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# Preface

This book was written originally in Dutch for Epsilon Uitgaven; the English version contains a number of corrections and some extensions, the largest of which are sections 11.6-7 and the exercises.

In writing this book I have kept two points in mind. First I wanted to produce a text which bridges the gap between elementary courses in ordinary differential equations and the modern research literature in this field. Secondly to do justice to the theory of differential equations and dynamical systems, one should present *both* the qualitative and quantitative aspects.

Thanks are due to a number of people. A.H.P. van der Burgh read and commented upon the first version of the manuscript; also the contents of section 10.2 are mainly due to him.

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The typing and TEX-editing of the text was carefully done by Diana Balk.

Writing this book was a very enjoyable experience. I hope that some of this pleasure is transferred to the reader when reading the text.

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April 1989

Ferdinand Verhulst

# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
1.1	Definitions and notation . . . . .	1
1.2	Existence and uniqueness . . . . .	3
1.3	Gronwall's inequality . . . . .	5
<b>2</b>	<b>Autonomous equations</b>	<b>7</b>
2.1	Phase-space, orbits . . . . .	7
2.2	Critical points and linearisation . . . . .	10
2.3	Periodic solutions . . . . .	14
2.4	First integrals and integral manifolds . . . . .	16
2.5	Evolution of a volume element, Liouville's theorem . . . . .	22
2.6	Exercises . . . . .	24
<b>3</b>	<b>Critical points</b>	<b>27</b>
3.1	Two-dimensional linear systems . . . . .	27
3.2	Remarks on three-dimensional linear systems . . . . .	31
3.3	Critical points of nonlinear equations . . . . .	33
3.4	Exercises . . . . .	37
<b>4</b>	<b>Periodic solutions</b>	<b>39</b>
4.1	Bendixson's criterion . . . . .	39
4.2	Geometric auxiliaries, preparation for the Poincaré-Bendixson theorem	41
4.3	The Poincaré-Bendixson theorem . . . . .	45
4.4	Applications of the Poincaré-Bendixson theorem . . . . .	49
4.5	Periodic solutions in $\mathbf{R}^n$ . . . . .	55
4.6	Exercises . . . . .	60
<b>5</b>	<b>Introduction to the theory of stability</b>	<b>62</b>
5.1	Simple examples . . . . .	62
5.2	Stability of equilibrium solutions . . . . .	64
5.3	Stability of periodic solutions . . . . .	66
5.4	Linearisation . . . . .	70
5.5	Exercises . . . . .	71

<b>6</b>	<b>Linear equations</b>	<b>73</b>
6.1	Equations with constant coefficients . . . . .	73
6.2	Equations with coefficients which have a limit . . . . .	75
6.3	Equations with periodic coefficients . . . . .	80
6.4	Exercises . . . . .	85
<b>7</b>	<b>Stability by linearisation</b>	<b>88</b>
7.1	Asymptotic stability of the trivial solution . . . . .	88
7.2	Instability of the trivial solution . . . . .	93
7.3	Stability of periodic solutions of autonomous equations . . . . .	97
7.4	Exercises . . . . .	99
<b>8</b>	<b>Stability analysis by the direct method</b>	<b>101</b>
8.1	Introduction . . . . .	101
8.2	Lyapunov functions . . . . .	103
8.3	Hamiltonian systems and systems with first integrals . . . . .	108
8.4	Applications and examples . . . . .	113
8.5	Exercises . . . . .	114
<b>9</b>	<b>Introduction to perturbation theory</b>	<b>117</b>
9.1	Background and elementary examples . . . . .	117
9.2	Basic material . . . . .	120
9.3	Naïve expansion . . . . .	123
9.4	The Poincaré expansion theorem . . . . .	126
9.5	Exercises . . . . .	128
<b>10</b>	<b>The Poincaré-Lindstedt method</b>	<b>130</b>
10.1	Periodic solutions of autonomous second-order equations . . . . .	130
10.2	Approximation of periodic solutions on arbitrary long time-scales . . .	135
10.3	Periodic solutions of equations with forcing terms . . . . .	138
10.4	The existence of periodic solutions . . . . .	140
10.5	Exercises . . . . .	143
<b>11</b>	<b>The method of averaging</b>	<b>145</b>
11.1	Introduction . . . . .	145
11.2	The Lagrange standard form . . . . .	148
11.3	Averaging in the periodic case . . . . .	149
11.4	Averaging in the general case . . . . .	154
11.5	Adiabatic invariants . . . . .	157
11.6	Averaging over one angle, resonance manifolds . . . . .	161
11.7	Averaging over more than one angle, an introduction . . . . .	165
11.8	Periodic solutions . . . . .	168
11.9	Exercises . . . . .	172

<b>12 Relaxation oscillations</b>	<b>177</b>
12.1 Introduction . . . . .	177
12.2 The van der Pol-equation . . . . .	178
12.3 The Volterra-Lotka equations . . . . .	180
<b>13 Bifurcation theory</b>	<b>183</b>
13.1 Introduction . . . . .	183
13.2 Normalisation . . . . .	185
13.3 Averaging and normalisation . . . . .	191
13.4 Centre manifolds . . . . .	193
13.5 Bifurcation of equilibrium solutions and Hopf bifurcation . . . . .	197
13.6 Exercises . . . . .	201
<b>14 Chaos</b>	<b>204</b>
14.1 The Lorenz-equations . . . . .	204
14.2 A mapping associated with the Lorenz-equations . . . . .	207
14.3 A mapping of $\mathbf{R}$ into $\mathbf{R}$ as a dynamical system . . . . .	209
14.4 Results for the quadratic mapping . . . . .	213
<b>15 Hamiltonian systems</b>	<b>218</b>
15.1 Summary of results obtained earlier . . . . .	218
15.2 A nonlinear example with two degrees of freedom . . . . .	220
15.3 The phenomenon of recurrence . . . . .	224
15.4 Periodic solutions . . . . .	226
15.5 Invariant tori and chaos . . . . .	227
15.6 The KAM theorem . . . . .	230
15.7 Exercises . . . . .	234
<b>Appendix 1: The Morse lemma</b>	<b>237</b>
<b>Appendix 2: Linear periodic equations with a small parameter</b>	<b>239</b>
<b>Appendix 3: Trigonometric formulas and averages</b>	<b>241</b>
<b>Answers and hints to the exercises</b>	<b>242</b>
<b>References</b>	<b>271</b>
<b>Index</b>	<b>275</b>