

Ferdinand Verhulst

Nonlinear Differential Equations and Dynamical Systems

With 127 Figures

Second, Revised and Expanded Edition



Springer

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