

VARIATIONAL ANALYSIS IN SOBOLEV AND *BV* SPACES *Applications to PDEs and Optimization* SECOND EDITION

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Contents

Preface to the Second Edition	ix
Preface to the First Edition	xi
1 Introduction	1
I Basic Variational Principles	5
2 Weak solution methods in variational analysis	7
2.1 The Dirichlet problem: Historical presentation	7
2.2 Test functions and distribution theory	14
2.3 Weak solutions	30
2.4 Weak topologies and weak convergences	39
3 Abstract variational principles	65
3.1 The Lax–Milgram theorem and the Galerkin method	65
3.2 Minimization problems: The topological approach	74
3.3 Convex minimization theorems	87
3.4 Ekeland’s ε -variational principle	94
4 Complements on measure theory	105
4.1 Hausdorff measures and Hausdorff dimension	105
4.2 Set functions and duality approach to Borel measures	119
4.3 Introduction to Young measures	132
5 Sobolev spaces	145
5.1 Sobolev spaces: Definition, density results	146
5.2 The topological dual of $H_0^1(\Omega)$. The space $H^{-1}(\Omega)$	159
5.3 Poincaré inequality and Rellich–Kondrakov theorem in $W_0^{1,p}(\Omega)$. .	161
5.4 Extension operators from $W^{1,p}(\Omega)$ into $W^{1,p}(\mathbf{R}^N)$. Poincaré inequalities and the Rellich–Kondrakov theorem in $W^{1,p}(\Omega)$	167
5.5 The Fourier approach to Sobolev spaces. The space $H^s(\Omega)$, $s \in \mathbf{R}$. .	173
5.6 Trace theory for $W^{1,p}(\Omega)$ spaces	178
5.7 Sobolev embedding theorems	184
5.8 Capacity theory and elements of potential theory	197
6 Variational problems: Some classical examples	219
6.1 The Dirichlet problem	220

6.2	The Neumann problem	226
6.3	Mixed Dirichlet–Neumann problems	236
6.4	Heterogeneous media: Transmission conditions	241
6.5	Linear elliptic operators	246
6.6	The linearized elasticity system	249
6.7	Introduction to the Signorini problem	258
6.8	The Stokes system	261
6.9	Convection-diffusion equations	264
6.10	Semilinear equations	267
6.11	The nonlinear Laplacian Δ_p	275
6.12	The obstacle problem	279
7	The finite element method	285
7.1	The Galerkin method: Further results	285
7.2	Description of finite element methods	287
7.3	An example	290
7.4	Convergence of the finite element method	291
7.5	Complements	303
8	Spectral analysis of the Laplacian	307
8.1	Introduction	307
8.2	The Laplace–Dirichlet operator: Functional setting	309
8.3	Existence of a Hilbertian basis of eigenvectors of the Laplace–Dirichlet operator	313
8.4	The Courant–Fisher min-max and max-min formulas	317
8.5	Multiplicity and asymptotic properties of the eigenvalues of the Laplace–Dirichlet operator	324
8.6	A general abstract theory for spectral analysis of elliptic boundary value problems	329
9	Convex duality and optimization	333
9.1	Dual representation of convex sets	333
9.2	Passing from sets to functions: Elements of epigraphical calculus	337
9.3	Legendre–Fenchel transform	343
9.4	Legendre–Fenchel calculus	352
9.5	Subdifferential calculus for convex functions	355
9.6	Mathematical programming: Multipliers and duality	364
9.7	A general approach to duality in convex optimization	380
9.8	Duality in the calculus of variations: First examples	387
II	Advanced Variational Analysis	391
10	Spaces BV and SBV	393
10.1	The space $BV(\Omega)$: Definition, convergences, and approximation	393
10.2	The trace operator, the Green’s formula, and its consequences	400
10.3	The coarea formula and the structure of BV functions	408
10.4	Structure of the gradient of BV functions	427
10.5	The space $SBV(\Omega)$	429

11	Relaxation in Sobolev, BV, and Young measures spaces	437
11.1	Relaxation in abstract metrizable spaces	437
11.2	Relaxation of integral functionals with domain $W^{1,p}(\Omega, \mathbf{R}^m)$, $p > 1$	440
11.3	Relaxation of integral functionals with domain $W^{1,1}(\Omega, \mathbf{R}^m)$	456
11.4	Relaxation in the space of Young measures in nonlinear elasticity	468
11.5	Mass transportation problems	480
12	Γ-convergence and applications	487
12.1	Γ -convergence in abstract metrizable spaces	487
12.2	Application to the nonlinear membrane model	491
12.3	Application to homogenization of composite media	496
12.4	Stochastic homogenization	506
12.5	Application to image segmentation and phase transitions	533
13	Integral functionals of the calculus of variations	547
13.1	Lower semicontinuity in the scalar case	547
13.2	Lower semicontinuity in the vectorial case	552
13.3	Lower semicontinuity for functionals defined on the space of measures	559
13.4	Functionals with linear growth: Lower semicontinuity in BV and SBV	562
14	Application in mechanics and computer vision	569
14.1	Problems in pseudoplasticity	569
14.2	Some variational models in fracture mechanics	576
14.3	The Mumford–Shah model	595
15	Variational problems with a lack of coercivity	599
15.1	Convex minimization problems and recession functions	599
15.2	Nonconvex minimization problems and topological recession	617
15.3	Some examples	625
15.4	Limit analysis problems	632
16	An introduction to shape optimization problems	643
16.1	The isoperimetric problem	644
16.2	The Newton problem	645
16.3	Optimal Dirichlet free boundary problems	648
16.4	Optimal distribution of two conductors	651
16.5	Optimal potentials for elliptic operators	654
17	Gradient flows	663
17.1	The classical continuous steepest descent	663
17.2	The gradient flow associated to a convex potential	670
17.3	Gradient flow associated to a tame function. Kurdyka–Łojasiewicz theory	724
17.4	Sequences of gradient flow problems	734
17.5	Steepest descent and gradient flow on general metric spaces	766
17.6	Minimizing movements and the implicit Euler scheme	768
	Bibliography	771
	Index	791