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Linear Algebra

An Introduction to Abstract Mathematics

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To my father,
who through his own example
taught me how to work,
and to my mother
in loving memory

Preface

These lecture notes, which took shape from a one-semester course taught to sophomores and juniors at Claremont McKenna College, constitute a substantial, abstract introduction to linear algebra. Although we use material from elementary calculus on occasion to illustrate major ideas, there are virtually no formal prerequisites. This is not to say that the material is easy. Many students who have never needed to make much effort in previous mathematics courses will find themselves seriously challenged.

What is the nature of linear algebra? One might give two antipodal and complementary replies; like wave and particle physics, both illuminate the truth:

> THE STRUCTURAL REPLY. Linear algebra is the study of vector spaces and linear transformations. A vector space is a structure which abstracts and generalizes certain familiar notions of both geometry and algebra. A linear transformation is a function between vector spaces that preserves elements of this structure. In some sense, this discipline allows us to import some long-familiar and well understood ideas of geometry into settings which are not geometric in any obvious way.

> THE COMPUTATIONAL REPLY. Linear algebra is the study of linear systems and, in particular, of certain techniques of matrix algebra that arise in connection with such systems. The aims of the discipline are largely computational, and the computations are indeed complex.

This text leans heavily, even dogmatically, toward the structural reply. In my experience in both pure and applied mathematics, the recognition that a given problem space is a vector space is often in itself of more value than any associated computation. Moreover, realistic computational problems are almost exclusively done by computer, and therefore incessant hand-drilling in matrix techniques is both redundant and maladroit. Finally, linear algebra as abstract, noncomputational mathematics is often one's first encounter with mathematics as understood and appreciated by mathematicians. I hope that the student will learn here that calculation is neither the principal mode, nor the principal goal, nor the principal joy of mathematics.

Throughout, we emphasize structure and concept over calculation. The lecturer will note that the first three chapters bear the names of fundamental categories. Here are some further examples:

(i) Early on, we explicitly introduce and use the language and most basic results of group theory.

(ii) A few diagram chases are suggested, one of which yields a particularly elegant proof of the change of basis formula.

(iii) The proof of the associative law for matrix multiplication is postponed until after the representation of linear transformations by matrices is introduced. At this point it becomes simply an aspect of the associativity of composition of functions.

(iv) The equality of the column and row ranks of a matrix is demonstrated entirely through the formal properties of the dual space. Accordingly, we define the transpose not just for matrices, but for arbitrary linear transformations, and we fully reconcile these two notions.

An outline of the exposition follows.

(1) Sets and Functions. We begin by reviewing notation and terminology, most of which should be familiar, in one form or another, from early calculus. The equivalence of bijectivity and invertibility of functions is the main result of the first half of the chapter. A brief digression on cardinality follows; this is not used at all in the sequel, but does provide a brief and appealing respite from the stream of formalities. The chapter concludes with an introduction to the symmetric group on n letters. This material is, of course, used later in the discussion of determinants and leads gracefully into the next, more radical topic.

(2) Groups and Group Homomorphisms. Abstract groups are admittedly nonstandard fare for a course in linear algebra, but the rationale for their inclusion here is, perhaps ironically, as much pedagogical as mathematical. On the mathematical side, a vector space is first an additive group, and a linear transformation is first a homomorphism of additive groups. Moreover, group theory plays a critical role in matrix theory. Thus we lay the foundations here for most of what follows, and what follows is made simpler thereby.[1] On the pedagogical side, one must recognize that the comprehension and composition of proofs is the central means by which one encompasses abstract concepts. The group, in all of its axiomatic austerity, provides a wonderful training ground for dealing with informal axiomatic systems. The student will at first lack intui-

[1]We do not venture beyond that which is mathematically prerequisite to the remainder of the text; for example, the basic counting theorems for finite groups are not included.

tion, but in learning to write correct proofs, this is not altogether a disadvantage. There are three sections. The first defines groups and subgroups and develops a few fundamental properties. The second introduces group homomorphisms, and already many linear algebraic themes begin to appear. (For instance, the characterization of the inverse image of an element under a homomorphism.) The last section briefly discusses rings and fields. The terminology is used throughout (rings sparingly), but the reader who takes *field* to mean either the real or complex numbers, will suffer no serious consequences.

(3) VECTOR SPACES AND LINEAR TRANSFORMATIONS. Building on additive group theory, this chapter introduces the two central objects of linear algebra, with some key examples. (The lecturer will no doubt want to supplement the material with many pictures.) We develop the basic arithmetic properties of vector spaces and subspaces, the notions of span and spanning sets, and the fundamental properties of linear transformations. Throughout, the proofs remain straightforward, and against the background of the previous chapter most students find little difficulty here.

(4) DIMENSION. The topics covered include linear dependence and its characterizations, basis and its *many* characterizations, and the fundamental structure theory of vector spaces: that every vector space admits a basis (*Vector spaces are free!*) and that every basis of a given space has the same cardinality. The student will see explicitly that the vector space axioms capture two primary concepts of geometry: coordinate systems and dimension. The chapter concludes with the Rank-Nullity Theorem, a powerful computational tool in the analysis of linear systems.

(5) MATRICES. Within this chapter, we move from matrices as arrays of numbers with a bizarre multiplication law to matrices as representations of linear systems and examples *par excellence* of linear transformations. We demonstrate but do not dwell on Gauss-Jordan Elimination and *LU* Decomposition as primary solution techniques for linear systems.

(6) REPRESENTATION OF LINEAR TRANSFORMATIONS. This long and difficult chapter, which establishes a full and explicit correspondence between matrices and linear transformations of finite-dimensional vector spaces, is the heart of the text. In some sense it justifies the structural reply to those who would compute, and the computational reply to those who would build theories. We first analyze the algebra of linear transformations on familiar spaces such as $\mathbf{R}^n$ and then pass to arbitrary finite-dimensional vector spaces. Here we present the momentous idea of the matrix of a transformation relative to a pair of bases and the isomorphism of algebras that this engenders. For the more daring, a discussion of the dual space follows, culminating in a wholly noncomputational and genuinely illuminating proof that the column and row ranks of a matrix are equal. Finally, we discuss transition matrices (first formally, then computa-

tionally), similarity of matrices, and the change of basis formula for endomorphisms of finite-dimensional spaces.

(7) INNER PRODUCT SPACES. The emphasis here is on how the additional structure of an inner product allows us to extend the notions of length and angle to an abstract vector space through the Cauchy-Schwarz Inequality. Orthogonal projection, the Gram-Schmidt process, and orthogonal complementation are all treated, at first for real inner product spaces, with the results later extended to the complex case. The examples and exercises make the connection with Fourier Series, although we only consider finite approximations.

(8) DETERMINANTS. The determinant is characterized by three fundamental properties from which all further properties (including uniqueness) are derived. The discussion is fairly brief, with only enough calculation to reinforce the main points. (Generally the student will have seen determinants of small matrices in calculus or physics.) The main result is the connection between the determinant and singularity of matrices.

(9) EIGENVALUES AND EIGENVECTORS. Virtually all of the threads are here woven together into a rich tapestry of surpassing texture. We begin with the basic definitions and properties of eigenvalues and eigenvectors and the determination of eigenvalues by the characteristic polynomial. We turn next to Hermitian and unitary operators and the orthogonality of their eigenspaces. Finally, we prove the Spectral Decomposition Theorem for such operators—one of the most delightful and powerful theorems in all of mathematics.

(10) TRIANGULATION AND DECOMPOSITION OF ENDOMORPHISMS. The discussion is a somewhat technical extension of the methods and results of the previous chapter and provides further insight into linear processes. (In a one-semester course, only the most energetic of lecturers and students is likely to alight on this turf.) In particular, we cover the Cayley-Hamilton Theorem, triangulation of endomorphisms, decomposition by characteristic subspaces, and reduction to the Jordan normal form.

With deep gratitude I wish to acknowledge the influence on this work of two of my teachers, both of whom are splendid mathematicians. Professor Wilfried Schmid taught me much of this material as an undergraduate. While my notes from his class have long since vanished, my impressions have not. Professor Hyman Bass, my thesis advisor at Columbia University, taught me whatever I know about writing mathematics. Many of his students have remarked that even his blackboards, without editing, are publishable! These men have a gift, unequaled in my experience, for the direct communication of mathematical aesthetics and mathematical experience. Let me also thank my own students at Claremont McKenna College for their patient and open-minded efforts to

encompass and to improve these notes. No one has ever had a more dedicated group.

Two of my best students, Julie Fiedler and Roy Corona, deserve special praise. While enrolled in my course, they proved to be such astute proofreaders that I asked them both to assist me through the final draft of the manuscript. Although I had expected that they would simply continue their proofreading, they quickly progressed beyond this, offering scores of cogent suggestions serving to clarify the material from the student's perspective. I hope that they recognize and take pride in the many changes their comments have effected in the final product.

The aesthetic affinity of mathematics and music has always been powerful—at least for mathematicians. At times we compose, and at times we conduct. Linear algebra is one of the great symphonies in the literature of mathematics, on a par with the large works of Beethoven, Brahms, or Schubert. And so without further delay, the conductor raises his baton, the members of the orchestra find their notes, and the music begins...

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SUPPLEMENTARY TOPICS

Index of Notation

Notation	*Section*	*Interpretation*
$\forall s$; $\exists s$	—	for all s; there exists an s
$p \Rightarrow q$	—	logical implication (if p, then q)
$p \Leftrightarrow q$	—	logical equivalence (p if and only if q)
$\varnothing$, $\cap$, $\cup$	—	null set, intersection, union, respectively
$S \subseteq T$	—	set inclusion (S is a subset of T)
$S \times T$	—	Cartesian product of sets
$\mathbf{N}$, $\mathbf{Z}$, $\mathbf{Q}$	—	natural numbers, integers, and rational numbers, respectively
$\mathbf{R}$, $\mathbf{C}$	—	real and complex numbers, respectively
$\mathbf{R}_+, \mathbf{R}_+^\times$	—	nonnegative reals, positive reals
$\mathbf{Q}^*, \mathbf{R}^*, \mathbf{C}^*$	—	nonzero elements of indicated set
$f: S \to T$	1.1	a function f with domain S and codomain T
$\mathrm{Im}(f)$	1.1	image of f
$1_S: S \to S$	1.1	identity function on S
$s \mapsto t$	1.1	s maps to t
$\mathscr{C}^0(\mathbf{R})$	1.1	continuous real-valued functions on $\mathbf{R}$
$g \circ f$	1.2	composition of functions
f^{-1}	1.3	inverse function
$\mathrm{Card}(S)$	1.4	cardinality of the set S
$\mathscr{P}(S)$	1.4	power set on S
S_n	1.5	symmetric group on n letters
$(a_1 \cdots a_k)$	1.5	k-cycle
$\sigma: S_n \to \{\pm 1\}$	1.5	sign homomorphism
$\mathbf{Z}_n$	2.1	integers modulo n
$G \cong G'$	2.2	G is isomorphic to G'
$f^{-1}(t)$	2.2	inverse image of an element