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Tullio Valent

Boundary Value Problems of Finite Elasticity

Local Theorems on Existence, Uniqueness, and Analytic Dependence on Data

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Tullio Valent
Dipartimento di Matematica Pura ed Applicata
Università di Padova
Padova 35131
Italy

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To Pia, Antonio, and Lucia

Preface

In this book I present, in a systematic form, some local theorems on existence, uniqueness, and analytic dependence on the load, which I have recently obtained for some types of boundary value problems of finite elasticity. Actually, these results concern an n-dimensional ($n \geqq 1$) formal generalization of three-dimensional elasticity. Such a generalization, besides being quite spontaneous, allows us to consider a great many interesting mathematical situations, and sometimes allows us to clarify certain aspects of the three-dimensional case. Part of the matter presented is unpublished; other arguments have been only partially published and in lesser generality. Note that I concentrate on simultaneous local existence and uniqueness; thus, I do not deal with the more general theory of existence. Moreover, I restrict my discussion to compressible elastic bodies and I do not treat unilateral problems. The clever use of the inverse function theorem in finite elasticity made by STOPPELLI [1954, 1957a, 1957b], in order to obtain local existence and uniqueness for the traction problem in hyperelasticity under dead loads, inspired many of the ideas which led to this monograph.

Chapter I aims to give a very brief introduction to some general concepts in the mathematical theory of elasticity, in order to show how the boundary value problems studied in the sequel arise.

Chapter II is very technical; it supplies the framework for all subsequent developments. Theorems on continuity, differentiability, and analyticity for composition operators are established in this chapter; they will suggest the later choices of the spaces for solutions and data. From

Theorem 6.1 it follows, for example, that to study our nonlinear problems using the implicit function theorem, the Sobolev spaces connected with a weak formulation of its (formally) linearized problems do not work. Thus we need appropriate regularity theorems for linear boundary value problems with a rather mild smoothness of some coefficients. The main object of Chapter III is precisely to provide such regularity theorems.

Subsequent chapters are devoted to obtaining theorems of existence, uniqueness, and analytic dependence on the load, near special deformations, for boundary value problems of place (in Chapter IV) and traction (in Chapters V and VI) in finite elastostatics. Loads independent of the deformation (dead loads) and loads depending on the deformation (live loads) are both considered. For the problem of place under dead loads some "semiglobal" results are also given. Evidently, a reasonable dependence of the load on the deformation, while not creating serious difficulties for the boundary condition of place, gives rise to a very wide variety of boundary value problems, with difficulties of every kind when we deal with the traction problem. On the other hand, for the traction problem, any physically realistic load depends (nontrivially) on the deformation.

In Chapter V, I present an abstract method of attacking the traction problem with general loads when certain conditions are satisfied: this method leads to an abstract theorem of existence, uniqueness, and analytic dependence on a parameter (Theorem 5.1). A first application of this theorem is given in the second part of Chapter V in treating the case of dead loads. Two more applications of that abstract theorem are made in Chapter VI, where a very important class of traction problems is studied: namely, those in which the prescribed surface traction is parallel to the normal of the boundary of the unknown deformed equilibrium configuration. Within this class there are boundary value problems to which the abstract method of Chapter V does not apply. One of these is particularly interesting and realistic: that is, the boundary value problem arising from the study of the equilibrium of a heavy elastic body submerged in a quiet, homogeneous, heavy liquid.

Most of Chapter VI is devoted to (the n-dimensional version of) this boundary value problem. The main result of the book is a theorem of existence, uniqueness, and analytic dependence on a parameter for this problem, near suitable deformations (see Theorem 4.17). I believe that some key ideas devised in proving this theorem may suggest a way of attacking boundary value problems of traction different from those discussed here. Moreover, I note that in traction problems, the particular deformations near to which I find existence, uniqueness, and analytic dependence on a parameter are unstressed; but, bearing in mind the analysis of Bharatha & Levinson [1978], Capriz & Podio-Guidugli [1979], and Wan & Marsden [1983], we can realize how the methodologies presented here can be adapted in order to study traction boundary value problems near stressed deformations.

Of course, the methods and results of this book have a quite different character from those based on the calculus of variations and the search for suitable constitutive assumptions (such as polyconvexity of the stored-energy function assumed by BALL [1977]). Rather, they may be useful as a first step in a global approach to boundary value problems of finite elasticity.

I conclude by expressing my gratitude to Professor C. TRUESDELL for inviting me to write this monograph.

Padova TULLIO VALENT
January 1987

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