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Multidimensional Weakly Singular Integral Equations

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In memory
of Solomon Mikhlin

PREFACE

In these lecture notes we deal with the integral equation

$$u(x) = \int_G K(x,y)\,u(y)dy + f(x), \quad x \in G, \tag{0.1}$$

where $G \subset \mathbb{R}^n$ is an open bounded region or, more generally, an open bounded set (possibly non-connected). The functions f and K are assumed to be smooth but K may have a weak singularity on the diagonal:

$$|K(x,y)| \le b(1 + |x-y|^{-\nu}), \qquad b = \text{const}, \quad \nu < n. \tag{0.2}$$

The main problems of interest to us are the following:

— the smoothness of the exact solution to equation (0.1);

— discretization methods for equation (0.1).

Usually, the derivatives of the solution to a weakly singular integral equation have singularities near the boundary ∂G of the domain of integration $G \subset \mathbb{R}^n$. A unified description of the singularities in all possible cases is complicated, and up to now this problem has not been solved fully. In Chapter 3 we give estimates which are sharp in many practically interesting cases. The behaviour of the tangential derivatives thereby turns out to be less singular than the behaviour of the normal derivatives. All this information is used designing approximate methods for integral equation (0.1). We restrict ourselves to collocation and related schemes, thoroughly examining simplest schemes based on the piecewise constant approximation of the solution and the superconvergence phenomenon at the collocation points (Chapters 5 and 6). In the case where $G \subset \mathbb{R}^n$ is a parallelepided, higher order collocation methods on graded grids are also considered (Chapter 7); again the superconvergence at the collocation points is examined.

Technically, our convergence analysis is based on the discrete convergence theory outlined in Chapter 4 of the book. This short chapter can be used for a first acquaintance with the theory for linear equations $u = Tu + f$; for eigenvalue problems and nonlinear equations, the results are presented without proofs.

In Chapter 8, some of the main results of Chapters 3 and 5-7 are extended to nonlinear integral equations.

Examples of (linear) integral equations (0.1), (0.2) can be found in radiation transfer theory (see Section 1); some interior-exterior boundary value problems too have their most natural formulations as integral equations of type (0.1), (0.2). Perhaps some readers will be disappointed to find that our treatment concerns only integral equations on an open set $G \subset \mathbb{R}^n$. In practice, there is a great interest also in the boundary integral equations

$$u(x) = \int_{\partial G} K(x,y)\,u(y)\,dS_y + f(x), \qquad x \in \partial G. \tag{0.3}$$

Such equations arise, for instance, in solving the Dirichlet or Neumann problem for the Laplace equation (see e.g. Mikhlin (1970) or Atkinson (1990)). A natural question of whether the results of the lecture book can be extended or modified to boundary integral equations then arises. The answer is non-unique. If ∂G is smooth then the solution of the boundary integral equation is smooth too, and the results concerning the collocation and related methods can even be strengthened and the arguments can be simplified. On the other hand, if ∂G is non-smooth then the standard boundary integral operators, e.g. the ones corresponding to the Laplace equation, are non-compact, and our arguments fail fully. The case of an integral equation on a smooth (relative) region $\Gamma \subset \partial G$ with a smooth (relative) boundary $\partial\Gamma$ seems to be the most adequate case that can be treated by our arguments. But this assertion may be considered only as a conjecture not discussed anywhere.

We use only a minimum of references in the main text. Nevertheless, an extended commented bibliography is added. Young mathematicians looking for problems to work on will find a list of unsolved problems too. The lectures are based on the author's recent publications (see Vainikko (1990a,b), (1991a,b), (1992a,b), Vainikko and Pedas (1990)) but actually the results were elaborated during a much longer time lecturing at University of Tartu, the Technical University of Chemnitz and Colorado State University. A significant milestone for us was the booklet by Vainikko, Pedas and Uba (1984) concerning the one-dimensional case ($n=1$). In the present lectures, we always assume that $n \geq 2$.

CONTENTS