

Mathematical World

Volume 19

A Mathematical Gift, I

The interplay between
topology, functions,
geometry, and algebra

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