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Locally Convex Spaces and Linear Partial Differential Equations

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To Ursula

Introduction

It is hardly an exaggeration to say that, if the study of general topological vector spaces is justified at all, it is because of the needs of distribution and Linear PDE* theories (to which one may add the theory of convolution in spaces of holomorphic functions). The theorems based on TVS** theory are generally of the "foundation" type: they will often be statements of equivalence between, say, the existence – or the approximability – of solutions to an equation $\mathbf{P}u=v$, and certain more "formal" properties of the differential operator $\mathbf{P}$, for example that $\mathbf{P}$ be elliptic or hyperbolic, together with properties of the manifold X on which $\mathbf{P}$ is defined. The latter are generally geometric or topological, e.g. that X be $\mathbf{P}$-convex (Definition 20.1). Also, naturally, suitable conditions will have to be imposed upon the data, the v's, and upon the stock of possible solutions u. The effect of such theorems is to subdivide the study of an equation like $\mathbf{P}u=v$ into two quite different stages. In the first stage, we shall look for the relevant equivalences, and if none is already available in the literature, we shall try to establish them. The second stage will consist of checking if the "formal" or "geometric" conditions are satisfied. Each one of these phases requires specific techniques: checking of the formal or the geometrical conditions generally demands "hard analysis" methods, might for instance require the construction of a fundamental solution, or the proof of uniqueness in a Cauchy problem. The proof of the equivalences – the first step – will usually rely on "soft analysis", that is, on the study of rather poor structures, such as those of some brand of locally convex spaces.

The present book is concerned with the soft analysis, applied to linear PDE's. It is essentially expository, and does not contain any new result on the subject of partial differential equations. Nevertheless, it has seemed to me that there was still some room, among the publications on this subject, for a short monograph, providing the statements and the proofs of most existence and approximation theorems in the field – furthermore,

* Throughout the book, PDE will stand for "partial differential equation".

** TVS will stand for "topological vector spaces".

providing them in a general, hence flexible, form, disengaged from technicalities – as much as it could be done.

However, the usefulness of such a book, which is at any rate moderate, would not have seemed to me a sufficient reason for its writing. The determining reason lies elsewhere: namely, in the possibility of giving a new, and greatly simplifying, presentation of the basic functional analysis. Teaching of the subject before many, and widely different, audiences had convinced me that some attempt at simplifying this presentation should be made. Lately, and gradually, the ways and means of such a simplification have emerged. They go surprisingly far. But let me try to convince the reader that some improvement was needed, if only from the pedagogical point of view. A good example is provided by the following two theorems, due to S. BANACH, often applied to PDE theory:

Theorem I. – *Let E, F be two Fréchet spaces, $u: E \to F$ a continuous linear map. The mapping u is surjective* (i.e., onto) *if and only if its transpose ${}^t u$: $F' \to E'$ is injective and has a weakly closed image* (i.e., range).

Theorem II. – *Let E be a Fréchet space, M' a linear subspace* (or a convex subset) *of the dual E' of E. In order that M' be weakly closed, it is necessary and sufficient that the intersection of M' with any equicontinuous subset H' of E' be weakly closed in H'.*

The proof of Theorem I makes use of the open mapping theorem, of the general fact that u is a weak homomorphism of E onto its image, $\operatorname{Im} u \subset F$, if and only if $\operatorname{Im} {}^t u$ is weakly closed in E', of the special fact that, for *metrizable* locally convex spaces (here, $E/\operatorname{Ker} u$ and $\operatorname{Im} u$) weak and "strong" isomorphisms are one and the same thing. The last assertion follows essentially from the fact that, in a metrizable locally convex space, a closed convex circled subset of E is a neighborhood of O if and only if it "swallows" any weakly convergent sequence. Careful study must be made of weak topologies, especially of the weak topologies on quotient spaces.

The key lemma, in proving Theorem II, is rather tricky. It applies again to a metrizable locally convex space E. It asserts that a subset U' of the dual E' of E is open, in the sense of the topology of uniform convergence on the compact subsets of E, if and only if its intersection $U' \cap H'$ with arbitrary equicontinuous subsets H' of E' is *weakly* open. Then one exploits the following results: (i) if E is metrizable and complete, i.e., if E is a Fréchet space, the topology of uniform convergence on the compact subsets of E is identical with the topology of uniform convergence on the compact *and convex* subsets of E; (ii) when E' carries the latter, its dual is (canonically) identical with E, hence, the closure of any convex subset of E' in this topology is the same as its weak closure.*

Both proofs have, in common, the feature that they involve, at some point, three different topologies – either on the duals, or on certain quotient spaces of the duals. The propensity of TVS theory to compare and to deal with several distinct topologies at the same time, on the same underlying space, this is what nonspecialists and students resent most – not unreasonably. For it is not unreasonable to expect that a reputedly semitrivial theory be of easy access.

Our exposition, from beginning to end, allows no topology to interfere, other than the one initially given on the space E under study. The topology of E is "embodied" in the set of all the continuous seminorms on E. This set is denoted by Spec E and called *spectrum* of E – for want of a better word.** It carries no topology: its natural structure of convex cone, and the properties of the order relation between nonnegative functions, here the seminorms, will suffice for our needs. The spectrum of E contains the total information we have about E. There is no reason why we should not take full advantage of it and why, during the proofs, we should limit ourselves to dealing only with those special seminorms, the absolute values of the linear functionals. In this manner, one can go rather far, as shown in Chapters I to IV. Chapter V, which is the last one in Part I, devoted to abstract functional analysis, describes the role left to duality. For there is a role left to duality, and quite an important one! Indeed, the study carried through from Chapter I to Chapter IV leads to the two main theorems of Part I – the *epimorphism theorem* and the *theorem of existence and approximation of solutions* to a functional equation. Both theorems are concerned with a linear map $u: E \to F$ and state, under suitable hypotheses, necessary and sufficient conditions in order that the equation $u(x) = y$ be "solvable" or in order that its solutions be approximated by solutions of a simpler kind. The conditions in question bear on general (continuous) seminorms, defined either in E or in F. But when the proofs of the general theorems are completed, when we come to checking if the conditions are satisfied, then we wish to be allowed to deal with the smallest possible set of seminorms, precisely with the seminorms, absolute value of a linear functional. True, it is not only because they are fewer that these are the natural candidates to the role of "test seminorms".

* Theorem I will be proved in Section 17; the proof given will be different from the one just outlined. Theorem II will never be applied and will not be proved.

** Actually, *speculum* = mirror would have been a better interpretation of Spec.

A more compelling reason is that distribution theory and the tools of harder analysis, such as Fourier-Laplace transformation, derivation of a priori estimates, pseudodifferential operators, etc., enable us to investigate in great depth the properties of linear functionals (generally, these are distributions of certain types). While, at least for the present, we are quite disarmed when faced with general convex functionals. The last section of Part I, n° 17, gives the "duality translation" (Theorems 17.1 and 17.2) of the epimorphism theorem and of the theorem on existence and approximation of solutions. The possibility of such a translation is based on one hand on the Hahn-Banach theorem, which was to be expected, essentially on the version that says that *a seminorm is the upper enveloppe of the absolute values of the linear functionals which are at most equal to it*, on the other hand, on Mackey's theorem, stating that *a seminorm* $\mathfrak{p}$ *is bounded on a set* $B \subset E$ *if and only if every linear functional* x' *on* E*, such that* $|x'| \leqslant \mathfrak{p}$*, is bounded on the set* B.

Part II presents the most important applications of Theorems 17.1 and 17.2 to linear PDE's. In most cases, the applications require no more introduction that the definition of the differential operators and of the functions and distributions spaces on which they act. For there lies another advantage of the "theory" developed in Part I: that the abstract theorems are much closer to their concrete forms than were their more classical counterparts, like Theorems I and II above: a priori estimates, semiglobal solvability (Definition 20.2), **P**-convexity (Definition 20.1) make sense, and very naturally, in the abstract set-up (see Definitions 11.1 and 11.2). One sees thus that the theorems about partial differential equations, presented in Part II, are really results of pure, and very simple, functional analysis.

In Part I, the only result which was admitted without a proof was the Hahn-Banach theorem. In Part II, I have felt free to use, without proving them, a certain number of facts about partial differential equations (or about analytic functions and analytic functionals), referring the reader to the existing literature, mainly to the books of HÖRMANDER [*1*] and of TREVES [*1*].

Certain topics, closely related to the material discussed in Part II, have been omitted. This is true of the numerous applications of Theorem 17.2 to the theory of "differential equations of infinite order", i.e., of convolution in spaces of holomorphic functions. They are straightforward applications and duplicate, in somewhat more involved situations, what is done for a linear partial differential operator acting on entire functions

(Theorem 29.1) or on holomorphic functions in convex open subsets of the complex space $\mathbf{C}^n$ (Theorem 30.2). Another important result which will not be found here is the theorem on existence of solutions to a linear PDE with constant coefficients, in the space of *all* distributions. It is due to L. Hörmander ([*1*], Theorem 3.6.4). The sufficiency part of this result is easy to establish in purely abstract terms, but I have not succeeded in finding *necessary and sufficient* conditions that generalize those given by Hörmander. The knowledge of such conditions would clarify the variable coefficients case.

As it stands, Part II of the present book is hardly more than an evolved version of Chapter I of Malgrange's thesis [*1*]. Of course, since 1955, some gain has been made in generality; more facts are known – the general picture is clearer.

I have added, at the end of the book, a summary of the definitions and the results concerning the spectrum of a locally convex space, that is, a summary of Part I. In this summary, no proofs are given. But since the proofs are all very simple, reading of it should provide a fairly accurate idea of what it is all about. Furthermore, the reader not too familiar with PDE theory will find, also at the end of the book, a very short dictionary, listing and explaining the definitions most often used about PDE (such as elliptic equation, fundamental solution, etc.).

Paris, December 1966. F. Treves

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