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978-0-521-87727-5 - Numerical Solution of Hyperbolic Partial Differential Equations

John A. Trangenstein

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