Texts in Applied Mathematics 33

Springer Science+Business Media, LLC

Texts in Applied Mathematics

1. *Sirovich:* Introduction to Applied Mathematics.
2. *Wiggins:* Introduction to Applied Nonlinear Dynamical Systems and Chaos.
3. *Hale/Koçak:* Dynamics and Bifurcations.
4. *Chorin/Marsden:* A Mathematical Introduction to Fluid Mechanics, 3rd ed.
5. *Hubbard/West:* Differential Equations: A Dynamical Systems Approach: Ordinary Differential Equations.
6. *Sontag:* Mathematical Control Theory: Deterministic Finite Dimensional Systems, 2nd ed.
7. *Perko:* Differential Equations and Dynamical Systems, 2nd ed.
8. *Seaborn:* Hypergeometric Functions and Their Applications.
9. *Pipkin:* A Course on Integral Equations.
10. *Hoppensteadt/Peskin:* Mathematics in Medicine and the Life Sciences.
11. *Braun:* Differential Equations and Their Applications, 4th ed.
12. *Stoer/Bulirsch:* Introduction to Numerical Analysis, 2nd ed.
13. *Renardy/Rogers:* A First Graduate Course in Partial Differential Equations.
14. *Banks:* Growth and Diffusion Phenomena: Mathematical Frameworks and Applications.
15. *Brenner/Scott:* The Mathematical Theory of Finite Element Methods.
16. *Van de Velde:* Concurrent Scientific Computing.
17. *Marsden/Ratiu:* Introduction to Mechanics and Symmetry, 2nd ed.
18. *Hubbard/West:* Differential Equations: A Dynamical Systems Approach: Higher-Dimensional Systems.
19. *Kaplan/Glass:* Understanding Nonlinear Dynamics.
20. *Holmes:* Introduction to Perturbation Methods.
21. *Curtain/Zwart:* An Introduction to Infinite-Dimensional Linear Systems Theory.
22. *Thomas:* Numerical Partial Differential Equations: Finite Difference Methods.
23. *Taylor:* Partial Differential Equations: Basic Theory.
24. *Merkin:* Introduction to the Theory of Stability of Motion.
25. *Naber:* Topology, Geometry, and Gauge Fields: Foundations.
26. *Polderman/Willems:* Introduction to Mathematical Systems Theory: A Behavioral Approach.
27. *Reddy:* Introductory Functional Analysis with Applications to Boundary-Value Problems and Finite Elements.
28. *Gustafson/Wilcox:* Analytical and Computational Methods of Advanced Engineering Mathematics.
29. *Tveito/Winther:* Introduction to Partial Differential Equations: A Computational Approach.
30. *Gasquet/Witomski:* Fourier Analysis and Applications: Filtering, Numerical Computation, Wavelets.
31. *Brémaud:* Markov Chains: Gibbs Fields, Monte Carlo Simulation, and Queues.
32. *Durran:* Numerical Methods for Wave Equations in Geophysical Fluid Dynamics.
33. *Thomas:* Numerical Partial Differential Equations: Conservation Laws and Elliptic Equations.

J.W. Thomas

Numerical Partial Differential Equations

Conservation Laws and Elliptic Equations

With 99 Illustrations

Springer

J.W. Thomas
Department of Mathematics
Colorado State University
Fort Collins, CO 80543
USA

Series Editors

Mathematics Subject Classification (1991): 65-01, 65M06, 65N06

Library of Congress Cataloging-in-Publication Data
Thomas J.W. (James William), 1941–
Numerical partial differential equations : finite difference methods / J.W. Thomas
p. cm. — (Texts in applied mathematics ; 22)
Includes bibliographical references and index.

1. Differential equations, Partial—Numerical solutions.
2. Finite differences. I. Title. II. Series.
QA377.T495 1995
315′.353—dc20 95-17143

Printed on acid-free paper.

Originally published by Springer-Verlag New York Inc. in 1999
Softcover reprint of the hardcover 1st edition 1999

Production managed by Terry Kornak; manufacturing supervised by Nancy Wu.
Photocomposed copy prepared by the author.

9 8 7 6 5 4 3 2 1

ISBN 978-1-4612-6821-5 ISBN 978-1-4612-0569-2 (eBook)
DOI 10.1007/978-1-4612-0569-2

This book is dedicated to some of the women in my life.

Susan, Shannon, Lindsay

Britt, Stephanie, Karen, Lisa

Megan, Kristen, Lindsay, Amanda, Kirsten

Kelley, Rachel, Kaylee, Lauren, Lindsay

Sara, Brooke, Randi, Ashlee

Kayla, Danielle, Cara, Eryn, Lauren

Caitlin, Alysha, Alex, Tara, Maree, Elizabeth, Caitlin

Series Preface

Mathematics is playing an ever more important role in the physical and biological sciences, provoking a blurring of boundaries between scientific disciplines and a resurgence of interest in the modern as well as the classical techniques of applied mathematics. This renewal of interest, both in research and teaching, has led to the establishment of the series: *Texts in Applied Mathematics (TAM)*.

The development of new courses is a natural consequence of a high level of excitement on the research frontier as newer techniques, such as numerical and symbolic computer systems, dynamical systems, and chaos, mix with and reinforce the traditional methods of applied mathematics. Thus, the purpose of this textbook series is to meet the current and future needs of these advances and encourage the teaching of new courses.

TAM will publish textbooks suitable for use in advanced undergraduate and beginning graduate courses and will complement the *Applied Mathematical Sciences (AMS)* series, which will focus on advanced textbooks and research level monographs.

Preface

This book is the second part of a two part text on the numerical solution of partial differential equations. Part 1 (*TAM 22: Numerical Partial Differential Equations: Finite Difference Methods*) is devoted to the basics and includes consistency, stability and convergence results for one and two dimensional parabolic and hyperbolic partial differential equations—both scalar equations and systems of equations. This volume, subtitled *Conservation Laws and Elliptic Equations*, includes stability results for difference schemes for solving initial–boundary–value problems, analytic and numerical results for both scalar and systems of conservation laws, numerical methods for elliptic equations and an introduction to methods for irregularly shaped regions and for computation problems that need grid refinements. In the Preface to Part 1 (included below), I describe the many ways that I have taught courses out of the material in both Parts 1 and 2. Although I will not repeat those descriptions here, I do emphasize that the two parts of the text are strongly intertwined. Part 1 was used to set up some of the material done in Part 2, and Part 2 uses many results from Part 1. The contribution that I hope Chapter 8 of Part 2 makes to the subject is to give a description of how to use the Gustafsson-Kreiss-Sundström-Osher theory (GKSO theory) to choose numerical boundary conditions when they are necessary. In Chapters 9 and 10 I try to give a reasonably complete coverage of numerical methods for solving conservation laws and elliptic equations. Chapter 11 is meant to introduce the reader to the fact that there are methods for treating irregular regions and for placing refined grids in regions that need them. It is hoped that Parts 1 and 2 help prepare the reader to solve a broad spectrum of problems involving partial differential equations.

In addition to the people I have already thanked in the Preface to Part 1, I would like to thank Ross Heikes who pushed me to include as much as I did in Chapter 9. I would also like to pay tribute to Amiram Harten. His papers, which were readable and of excellent quality, have made a large contribution to the field of the numerical solution of conservation laws. And finally, I would like to thank my family, Ann, David, Michael, Carrie and Susan, for the patience shown when Part 2 took me much longer to write than I predicted. As before, the mistakes are mine and I would appreciate it if you sould send any mistakes that you find to thomas@math.colostate.edu. Thank you.

J.W. Thomas

Preface to Part 1

This textbook is in two parts. The first part contains Chapters 1–7 and is subtitled *Finite Difference Methods.* The second part contains Chapters 8–11 and is subtitled *Conservation Laws and Elliptic Equations.* This text was developed from material presented in a year long, graduate course on using difference methods for the numerical solution of partial differential equations. Writing this text has been an iterative process, and much like the Jacobi iteration scheme presented in Chapter 10, convergence has been slow. The course at Colorado State University is designed for graduate students in both applied mathematics and engineering. The students are required to have at least one semester of partial differential equations and some programming capability. Generally, the classes include a broad spectrum of types of students, ranging from first year mathematics graduate students with almost no physical intuition into the types of problems we might solve, to third year engineering graduate students who have a lot of physical intuition and know what types of problems they personally want to solve and why they want to solve them. Since the students definitely help shape the class that is taught, they probably have also helped to shape this text.

There are several distinct goals of the courses. One definite goal is to prepare the students to be competent practitioners, capable of solving a large range of problems, evaluating numerical results and understanding how and why results might be bad. Another goal is to prepare the applied mathematics Ph.D. students to take additional courses (the third course in our sequence is a course in computational fluid dynamics which requires both semesters taught out of this text) and to write theses in applied mathematics.

One of the premises on which this text is based is that in order to understand the numerical solution of partial differential equations the student must solve partial differential equations. The text includes homework problems that implement different aspects of most of the schemes discussed. As a part of the implementation phase of the text, discussions are included on how to implement the various schemes. In later parts of the text, we return to earlier problems to discuss the results obtained (or that should have been obtained) and to explain why the students got the results they did. Throughout the text, the problems sometimes lead to bad numerical results. As I explain to my students, since these types of results are very common in the area of numerical solutions of partial differential equations, they must learn how to recognize them and deal with them. A point of emphasis in my course, which I hope that I convey also in the text, is teaching the students to become experimentalists. I explain that before one runs an experiment, one should know as much as possible about the problem. A complete problem usually includes the physical problem, the mathematical problem, the numerical scheme and the computer. (In this text, the physical problem is often slighted.) I then try to show how to run numerical experiments. As part of the training to be a numerical experimentalist, I include in the Prelude four nonlinear problems. I assume that the students do not generally know much about these problems initially. As we proceed in the text, I suggest that they try generalizations of some of our linear methods on these nonlinear problems. Of course, these methods are not always successful and in these cases I try to explain why we get the results that we get.

The implementation aspect of the text obviously includes a large amount of computing. Another aspect of computing included in the text is symbolic computing. When we introduce the concept of consistency, we show the calculations as being done on paper. However, after we have seen a few of these, we emphasize that a computer with a symbolic manipulator should be doing these computations. When we give algorithms for symbolic computations, we have tried to give it in a pseudo code that can be used by any of the symbolic manipulators. Another aspect of the new technologies that we use extensively is graphics. Of course, we provide plots of our numerical results and ask the students to provide plots of their results. We also use graphics for analysis. For example, for the analyses of dissipation and dispersion, where much of this has traditionally been done analytically (where one obtains only asymptotic results), we emphasize how easy it is to plot these results and interpret the dissipativity and dispersivity properties from the plots.

Though there is a strong emphasis in the text on implementing the schemes, there is also a strong emphasis on theory. Because of the audience, the theory is usually set in what might be called computational space (where the computations are or might be done) and the convergence is done in $\ell_{2,\Delta x}$ spaces in terms of the energy norm. Though at times these spaces might not be as nice mathematically as some other spaces that might be used, it seems that working in spaces that mimic the computational space is easier for the students to grasp. Throughout the text, we emphasize the meaning of consistency, stability and convergence. In my classes I emphasize that it is dangerous for a person who is using difference methods not to understand what it means for a scheme to converge. In my class and in the text, I emphasize that we sometimes get necessary and sufficient conditions for convergence and sometimes get only necessary conditions (then we must learn to accept that we have only necessary conditions and proceed with caution and numerical experimentation). In the text, not only do we prove the Lax Theorem, but we return to the proof to see how to choose an initialization scheme for multilevel schemes and how we can change the definition of stability when we consider higher order partial differential equations. For several topics (specifically for many results in Chapters 8, 9 and 10) we do not include all of the theory (specifically not all of the proofs) but discuss and present the material in a theoretically logical order. When theorems are used without proof, references are included.

Lastly, it is hoped that the text will become a reference book for the students. In the preparation of the text, I have tried to include as many aspects of the numerical solution of partial differential equations as possible. I do not have time to include some of these topics in my course and might not want to include them even if I had time. I feel that these topics must be available to the students so that they have a reference point when they are confronted with them. One such topic is the derivation of numerical schemes. I personally do not have a preference on whether a given numerical scheme is derived mathematically or based on some physical principles. I feel that it is important for the student to know that they can be derived both ways and that both ways can lead to good schemes and bad schemes. In Chapter 1, we begin by first deriving the basic difference mathematically, and then show how the same difference scheme can be derived by using the integral form of the conservation law. We emphasize in this section that the errors using the latter approach are errors in numerical integration. This is a topic that I discuss and that I want the students to know is there and that it is a possible approach. It is also a topic that I do not develop fully in my class. Throughout the text, we return to this approach to show how it differs when we have two dimensional problems, hyperbolic problems, etc. Also, throughout the text we derive difference schemes purely mathematically (heuristically, by the method of undetermined coefficients or by other methods). It is hoped the readers will understand that if they have to derive their own schemes for a partial differential equation not previously considered, they will know where to find some tools that they can use.

Because of the length of the text, as was stated earlier, the material is being given in two parts. The first part includes most of the basic material on time dependent equations including parabolic and hyperbolic problems, multi-dimensional problems, systems and dissipation and dispersion. The second part includes chapters on stability theory for initial–boundary value problems (the GKSO theory), numerical schemes for conservation laws, numerical solution of elliptic problems and an introduction to irregular regions and irregular grids. When I teach the course, I usually cover most of the first five chapters during the first semester. During the second semester I usually cover Chapters 6 and 7 (systems and dissipation and dispersion), Chapter 10 (elliptic equations) and selected topics from Chapters 8, 9 and 11. In other instances, I have covered Chapters 8 and 9 during the second semester, and on one occasion, I used a full semester to teach Chapter 9. Other people who have used the notes have covered parts of Chapters 1–7 and Chapter 10 in one semester. In either case, there seems to be sufficient material for at least two semesters of course work.

At the end of most of the chapters of the text and in the middle of several, we include sections which we refer to as "Computational Interludes." The original idea of these sections was to stop working on new methods, take a break from theory and compute for a while. These sections do include this aspect of the material, but as they developed, they also began to include more than just computational material. It is in these sections that we discuss results from previous homework problems. It is also in these sections that we suggest it is time for the students to try one of their new methods on one of the problems HW0.0.1–HW0.0.4 from the Prelude. There are also some topics included in these sections that did not find a home elsewhere. At times a more appropriate title for these sections might have been "etc.".

At this time I would like to acknowledge some people who have helped me with various aspects of this text. I thank Drs. Michael Kirby, Steve McKay, K. McArthur and K. Bowers for teaching parts of the text and providing me with feedback. I also thank Drs. Kirby, McArthur, Jay Bourland, Paul DuChateau and David Zachmann for many discussions about various aspects of the text. Finally, I thank the many students who over the years put up with the dreadfully slow convergence of this material from notes to text. Whatever the result, without their input the result would not be as good. And, finally, though all of the people mentioned above and others have tried to help me, there are surely still some typos and errors of thought (though, hopefully, many mistakes have been corrected for the Second Printing). Though I do so sadly, I take the blame for all of these mistakes. I would appreciate it if you would send any mistakes that you find to thomas@math.colostate.edu. Thank you.

J.W. Thomas

Contents

Contents of Part 1: Finite Difference Methods