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# **The Four-Color Theorem**

History, Topological Foundations,  
and Idea of Proof

Translated by Julie Peschke

With 42 Illustrations



**Springer**

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The cover illustration shows a map explaining the failure in Kempe's "proof" of the Four-Color Theorem. See page 176.

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