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The Fokker-Planck Equation

Methods of Solution and Applications

Second Edition

With 95 Figures



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