Lecture Notes of the Unione Matematica Italiana 1

Luc Tartar

An Introduction to Navier–Stokes Equation and Oceanography

Springer

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Author

Luc Tartar
Department of Mathematical Sciences
Carnegie Mellon University
Pittsburgh, PA 15213-3890
USA

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In memory of Jean LERAY, Nov. 7, 1906 – Nov. 10, 1998
In memory of Olga LADYZHENSKAYA, Mar. 7, 1922 – Jan. 12, 2004

They pioneered the mathematical study of the Navier–Stokes equation, which is an important part of these lecture notes.

In memory of my father
Georges TARTAR, Oct. 9, 1915 – Aug. 5, 2003

He dedicated his life to what he believed God expected from him. As for me, once the doubt had entered my mind, what other choice did it leave me but to search for the truth, in all fields?

To my children
Laure, Michaël, André, Marta

Preface

In the spring of 1999, I taught (at Carnegie Mellon University) a graduate course entitled *Partial Differential Equations Models in Oceanography*, and I wrote lecture notes which I distributed to the students; these notes were then made available on the Internet, and they were distributed to the participants of a Summer School held in Lisbon, Portugal, in July 1999. After a few years, I feel it will be useful to make the text available to a larger audience by publishing a revised version.

To an uninformed observer, it may seem that there is more interest in the Navier–Stokes equation nowadays, but many who claim to be interested show such a lack of knowledge about continuum mechanics that one may wonder about such a superficial attraction. Could one of the Clay Millennium Prizes be the reason behind this renewed interest? Reading the text of the conjectures to be solved for winning that particular prize leaves the impression that the subject was not chosen by people interested in continuum mechanics, as the selected questions have almost no physical content. Invariance by translation or scaling is mentioned, but why is invariance by rotations not pointed out and why is Galilean invariance[1] omitted, as it is the essential fact which makes

[1] Velocities involved for ordinary fluids being much smaller than the velocity of light c, no relativistic corrections are necessary and Galilean invariance should then be used, but one should be aware that once the mathematical equation has been written it is not automatic that its solutions will only use velocities bounded by c. One should learn to distinguish between a mathematical property of an equation and a conjecture that some property holds which one guesses from the belief that the equation corresponds to a physical problem. One should learn about which defects are already known concerning how a mathematical model describes physical reality, but one should not forget that a mathematical model which is considered obsolete from the physical point of view may still be useful for mathematical reasons. I often wonder why so many forget to mention the defects of the models that they study.

the equation introduced by NAVIER[2] much better than that introduced later by STOKES? If one had used the word "turbulence" to make the donator believe that he would be giving one million dollars away for an important realistic problem in continuum mechanics, why has attention been restricted to unrealistic domains without boundary (the whole space R^3, or a torus for periodic solutions), as if one did not know that vorticity is created at the boundary of the domain? The problems seem to have been chosen in the hope that they will be solved by specialists of harmonic analysis, and it has given the occasion to some of these specialists to help others in showing the techniques that they use, as in a recent book by Pierre Gilles LEMARIÉ-RIEUSSET [17]; some of the techniques are actually very similar to those that I have learnt in the theory of interpolation spaces, on which I have already written some lecture notes which I plan to revise, and I hope that this particular set of lecture notes on the Navier–Stokes equation and another one not yet finished on kinetic theory may help the readers understand a little more about the physical content of the equation, and also its limitations, which many do not seem to be aware of.

Being a mathematician interested in science, and having learnt more than most mathematicians about various aspects of mechanics and physics,[3] one reason for teaching various courses and writing lecture notes is to help isolated researchers to learn about some aspects unknown to most mathematicians whom they could meet, or read. A consequence of this choice is then to make researchers aware that some who claim to work on problems of continuum

[2] The equation which one calls now after both NAVIER and STOKES was introduced by NAVIER, while STOKES only later introduced the equation without inertial effects, which is linear and does not present so much mathematical difficulty nowadays, but there are cases where the nonlinear term in the Navier equation disappears and the equation reduces to the Stokes equation, an example being irrotational flows. If one believes that the Stokes equation is a good model for small velocities (and bounded derivatives of the velocity) then using the Stokes equation in a frame moving at a local velocity and invoking Galilean invariance makes one discover the Navier equation (which I shall call the Navier–Stokes equation apart from this footnote); of course, I shall point out other defects of the model along the way.

[3] Classical mechanics is an 18th century point of view of mechanics, which requires ordinary differential equations as mathematical tools. Continuum mechanics is an 18th –19th century point of view of mechanics, which requires partial differential equations as mathematical tools; the same is true for many aspects of physics. However, 20th century aspects of mechanics (plasticity, turbulence) or physics (quantum effects) require mathematical tools which are beyond partial differential equations, similar to those that I have tried to develop in my research work, improving concepts such as *homogenization*, *compensated compactness* and *H-measures*.

mechanics or physics have forgotten to point out known defects of the models that they use.[4]

I once heard my advisor, Jacques-Louis LIONS, mention that once the detailed plan of a book is made, the book is almost written, and he was certainly speaking of experience as he had already written a few books at the time. He gave me the impression that he could write directly a very reasonable text, which he gave to a secretary for typing; maybe he then gave chapters to one of his students, as he did with me for one of his books [19], and very few technical details had to be fixed. His philosophy seemed to be that there is no need to spend too much time polishing the text or finding the best possible statement, as the goal is to take many readers to the front of research, or to be more precise to one front of research, because in the beginning he changed topics every two or three years. As for myself, I have not yet written a book, and the main reason is that I am quite unable to write in advance a precise plan of what I am going to talk about, and I have never been very good at writing even in my mother tongue (French). When I write, I need to read again and again what I have already written until I find the text acceptable (and that notion of acceptability evolves with time and I am horrified by my style of twenty years ago), so this way of writing is quite inefficient, and makes writing a book prohibitively long. One solution would be not to write books, and when I go to a library I am amazed by the number of books which have been written on so many subjects, and which I have not read, because I never read much. Why then should I add a new book? However, I am even more amazed by the number of books which are not in the library, and although I have access to a good inter-library loan service myself, I became concerned with how difficult it is for isolated students to have access to scientific knowledge (and I do consider mathematics as part of science, of course).

I also thought of a different question. It is clear that fewer and fewer students in industrialized countries are interested in studying mathematics, for various reasons, and as a consequence more and more mathematicians are likely to come from developing countries. It will therefore be of utmost importance that developing countries should not simply become a reservoir of good students that industrialized countries would draw upon, but that these countries develop a sufficiently strong scientific environment for the benefit of their own economy and people, so that only a small proportion of the new trained generations of scientists would become interested in going to work abroad. I have seen the process of decolonization at work in the early 1960s, and I have witnessed the consequences of too hasty a transition, which was not to the benefit of the former colonies, and certainly the creation of a scientific tradition is not something that can be done very fast. I see the development

[4] Of course, I also suffer from the same disease of not having learnt enough, but my hope is that by explaining what I have already understood and by showing how to analyze and criticize classical models, many will acquire my understanding and a few will go much further than I have on the path of discovery.

of mathematics as a good way to start building a scientific infrastructure, and inside mathematics the fields that I have studied should play an important role, where mathematics interacts with continuum mechanics and physics.

In the spring of 1999, I found the right solution for me, which is to give a course and to prepare lecture notes for the students, trying to write down after each lecture the two or three pages describing what I have just taught; for such short texts my problems about writing are not too acute. I could hardly have guessed at the beginning of the course how much an introduction to oceanography my course would become, and when after a short introduction and the description of some classical methods for solving the Navier–Stokes equation (in the over-simplified version which mathematicians usually consider), it was time to describe some of the models considered in oceanography, I realized that I did not believe too much in the derivation of these models, and I preferred to finish the course by describing some of the general mathematical tools for studying the nonlinear partial differential equations of continuum mechanics, some of which I have developed myself. The resulting set of lecture notes is not as good as I would have liked, but an important point was to make this introductory course available on the Internet. In the spring of 2000, I wrote similar lecture notes for a course divided into two parts, the first part on Sobolev spaces, and the second part on the theory of interpolation spaces, and in the fall of 2001, I wrote lecture notes for an introduction to kinetic theory; of course, it is my plan to finish and review these lecture notes to make them more widely available by publishing them.

I decided at that time to add some information that one rarely finds in courses of mathematics, something about the people who have participated in the creation of the knowledge related to the subject of the course. I had the privilege to study in Paris in the late 1960s, to have great teachers like Laurent SCHWARTZ and Jacques-Louis LIONS, and to have met many famous mathematicians. This has given me a different view of mathematics than the one that comes from reading books and articles, which I find too dry, and I have tried to give a little more life to my story by telling something about the actors; for those mathematicians whom I have met, I have used their first names in the text, and I have tried to give some simple biographical data for all people quoted in the text, in order to situate them, both in time and in space. For mathematicians of the past, a large part of this information comes from using *The MacTutor History of Mathematics archive* (http://www-history.mcs.st-and.ac.uk/history), for which one should thank J. J. O'CONNOR and E. F. ROBERTSON, from the University of St Andrews in Scotland, UK, but for many other names I searched the Internet, and it is possible that some of my information is incomplete or even inaccurate. My interest in history is not recent, but my interest in the history of mathematics has increased recently, in part from finding the above-mentioned archive, but also as a result of seeing so many ideas badly attributed, and I have tried to learn more about the mathematicians who have introduced some of the ideas which I was taught when I was a student in Paris in the late 1960s, and be as accurate as possible concerning the work of all. I hope that I shall be given the correct information by anyone who finds one of my inaccuracies, and that I shall be forgiven for these unintentional errors.

I was born in France in December 1946 from a Syrian father and a French mother and I left France for political reasons, and since 1987 I have enjoyed the hospitality of an American university, CARNEGIE MELLON University, in Pittsburgh PA, but I am still a French citizen, and I only have resident status in United States. This may explain my interest in mentioning that others have worked in a different country than the one where they were born, and I want to convey the idea that the development of mathematics is an international endeavor, but I am not interested in the precise citizenship of the people mentioned, or if they feel more attached to the country were they were born or the one where they work; for example, I quote Olga OLEINIK as being born in Ukraine and having worked in Moscow, Russia, and obviously Ukraine was not an independent country when she was born, but was when she died; a French friend, Gérard TRONEL, has told me that she did feel more Russian than Ukrainian, but if I have been told that information about her I completely lack information about others. Because some countries have not always existed or have seen their boundaries change by their own expansion or that of other countries, some of my statements are anachronistic, like when I say that Leonardo DA VINCI was Italian, but I do not say that for ARCHIMEDES, who is known to have died at the hand of a Roman soldier, or decide about EUCLID, or AL KHWARIZMI, as it is not known where they were born.

I observe that there have been efficient schools in some areas of mathematics at some places and at some moments in time, and when I was a student in Paris in the late 1960s, Jacques-Louis LIONS had mentioned that Moscow was the only other place comparable to Paris for its concentration of mathematicians. Although the conditions might be less favorable outside important centers, I want to think that a lot of good work could be done elsewhere, and my desire is that my lecture notes may help isolated researchers participate more in the advance of scientific knowledge. A few years ago, an Italian friend, GianPietro DEL PIERO, told me that he had taught for a few months in Somalia, and he mentioned that one student had explained to him that he should not be upset if some of the students fell asleep during his lectures, because the reason was not their lack of interest in the course, but the fact that sometimes they had eaten nothing for a few days. It was by thinking about these courageous students who, despite the enormous difficulties that they encounter in their everyday life, are trying to acquire some precious knowledge about mathematics, that I devised my plan to write lecture notes and make them available to all, wishing that they could arrive freely to isolated students and researchers, working in much more difficult conditions than those having access to a good library, or in contact with good teachers. I hope that publishing this revised version will have the effect that it will reach many libraries scattered around the world, where isolated researchers have access.

I hope that my lack of organizational skills will not bother the readers too much. I consider teaching courses like leading groups of newcomers into countries which are often unknown to them, but not unknown to me, as I have often wandered around; some members of a group who have already read about the region or have been in other expeditions with guides more organized than me might feel disoriented by my choice of places to visit, and indeed I may have forgotten to show a few interesting places, but my goal is to familiarize the readers with the subject and encourage them to acquire an open and scientific point of view, and not to write a definitive account of the subject.

There are results which are repeated, but it is inevitable in a real course that one should often recall results which have already been mentioned. There are also results which are mentioned without proof, and sometimes they are proven later but sometimes they are not, and if no references are given, one should remember that I have been trained as a mathematician, and that my statements without proofs have indeed been proven in a mathematical sense, because if they had not I would have called them conjectures instead;[5] however, I am also human and my memory is not perfect and I may have made mistakes. I think that the right attitude in mathematics is to be able to explain all the statements that one makes, but in a course one has to assume that the reader already has some basic knowledge of mathematics, and some proofs of a more elementary nature are omitted. Here and there I mention a result that I have heard of, but for which I never read a proof or did not make up my own proof, and I usually say so. If many proofs are mine it does not necessarily mean that I was the first to prove the corresponding result, but that I am not aware of a prior proof, maybe because I never read much. Actually, my advisor mentioned to me that it is useful to read only

[5] Some people like to talk of pure mathematics versus applied mathematics, but I do not think that such a distinction is accurate, as I mentioned in the introduction of an article for a conference at École Polytechnique (Palaiseau, France) in the fall of 1983, but because that introduction was cut by political censors, it is worth repeating that for what concerns different parts of mathematics there are those which I know, those which I do not know well but think that they could be useful to me, and those which I do not know well but do not see how they could be useful to me, and all this evolves with time, so I finally wonder if it is reasonable to classify mathematics as being pure or applied. I consider myself as an “applied” mathematician, although I give it a French meaning (a mathematician interested in other fields of science), opposed to a British meaning (a specialist of continuum mechanics, allowed to use an incomplete mathematical proof without having to call the result a conjecture), and in French universities, applied mathematicians in the British style are found in departments of mécanique. Probably for the reason of funding, which strangely enough is given more easily to people who pretend to do applied research, some who have studied to become mathematicians practice the art of using words which make naive people wrongly believe that they know continuum mechanics or physics, and I find this attitude potentially dangerous for the university system.

the statement of a theorem and one should read the proof only if one cannot supply one.[6]

My personal mathematical training has been in functional analysis and partial differential equations, starting at École Polytechnique, Paris, France, where I had two great teachers, Laurent SCHWARTZ and Jacques-Louis LIONS. Having studied there in order to become an engineer, but having had to change my orientation once I had been told that such a career required administrative skills (which I lack completely), I opted for doing research in mathematics with an interest in other sciences and I asked Jacques-Louis LIONS to be my advisor, and it was normal that once I had been taught enough on the mathematical side, I would apply my improved understanding to investigating questions of continuum mechanics and physics which I had heard about as a student, and to developing the new mathematical tools which are necessary for that.

In my lectures I also try to teach mathematicians about the defects of the models used, but I want to apologize for some of the words which I use, which may have offended some. I have a great admiration for the achievements of physicists and engineers[7] during the last century, and a lot of the improvements in our lives result from their understanding, which is so different than the type of understanding that mathematicians are trained to achieve. If I write that something that they say does not make any sense, it is not a criticism towards physicists or engineers, who are following the rules of their profession, but it is a challenge to my fellow mathematicians that there is something there that mathematicians ought to clarify. I am grateful to Robert DAUTRAY[8] for offering me a position at Commissariat à l'Énergie Atomique from 1982 to 1987, and for helping me understand more about physics through his advice during these years; he helped me understand what the challenges

[6] The MacTutor archive mentions an interesting anecdote in this respect concerning a visit of Antoni ZYGMUND to the University of Buenos Aires, Argentina, in 1948; Alberto CALDERÓN was a student there and he was puzzled by a question that ZYGMUND had asked, and he said that the answer was in ZYGMUND's own book *Trigonometric Series*, but there was disagreement on this point; what had happened was that CALDERÓN had read a statement in the book and supplied his own proof, which was more general that the one written, so it also answered the question that ZYGMUND had just asked, but CALDERÓN had wrongly assumed that ZYGMUND's proof in his book, which he had never checked, was similar to his. Franco BREZZI mentioned to me that Ennio DE GIORGI had once told Claudio BAIOCCHI something similar, that he almost never read a proof, and that he did his own proofs for the interesting theorems but that he did not bother to think about the uninteresting ones.

[7] I am not mentioning biologists and chemists because biology was not part of my studies, and although I have learnt some chemistry, I only hope to understand it in a better way once my program for understanding continuum mechanics and physics has progressed enough.

[8] A good reference for learning classical mathematical tools and their use in problems of engineering or physics is the collection of books that Robert DAUTRAY had persuaded Jacques-Louis LIONS to edit with him, [4–9]

are, and I hope that through my lecture notes more will understand about the challenges, and that should make Science progress.

The support of a few friends gave me the strength to decide to complete the writing of some unfinished lecture notes and to revise those which I had already written, with a view to publishing them to attain a wider audience. I want to express my gratitude to Thérèse BRIFFOD, for her hospitality when I carried out the first revision of this course in August 2002, but also for her help in making me understand better an important question in life, having compassion for those who are in difficulty. I want to express my gratitude to Lucia OSTONI, for her hospitality when I carried out the second revision of this course in July 2004, and the final adjustments to Springer's formatting in December 2004.

I want to thank my good friends Carlo SBORDONE and Franco BREZZI for having proposed to publish my lecture notes in a series of Unione Matematica Italiana, and for having helped me to arrive at the necessary corrections of my original text.

Milano, December 2004

Luc TARTAR
Correspondant de l'Académie des Sciences, Paris
University Professor of Mathematics
Department of Mathematical Sciences
CARNEGIE MELLON University
Pittsburgh, PA 15213-3890, United States of America

Introduction

In teaching a mathematical course where the Navier[9]–Stokes[10,11] equation plays a role, one must mention the pioneering work of Jean LERAY[12,13] in the 1930s. Some of the problems that Jean LERAY left unanswered are still open today,[14] but some improvements were started by Olga LADYZHENSKAYA[15] [16], followed by a few others, like James SERRIN,[16] and my advisor, Jacques-Louis LIONS[17] [19], from whom I learnt the basic principles for the mathematical analysis of these equations in the late 1960s.

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[9] Claude Louis Marie Henri NAVIER, French mathematician, 1785–1836. He worked in Paris, France.

[10] Sir George Gabriel STOKES, Irish-born mathematician, 1819–1903. He held the Lucasian chair at Cambridge, England, UK.

[11] Henry LUCAS, English clergyman, 1610–1663.

[12] Jean LERAY, French mathematician, 1906–1998. He received the Wolf Prize in 1979. He held a chair (Théorie des équations différentielles et fonctionnelles) at Collège de France, Paris, France.

[13] Ricardo WOLF, German-born (Cuban) diplomat and philanthropist, 1887–1981. The Wolf Foundation was established in 1976 with his wife, Francisca SUBIRANA-WOLF, 1900–1981, *to promote science and art for the benefit of mankind*.

[14] Most problems are much too academic from the point of view of continuum mechanics, because the model used by Jean LERAY is too crude to be meaningful, and the difficulties of the open questions are merely of a technical mathematical nature. Also, Jean LERAY unfortunately called turbulent the weak solutions that he was seeking, and it must be stressed that turbulence is certainly not about regularity or lack of regularity of solutions, nor about letting time go to infinity either.

[15] Olga Aleksandrovna LADYZHENSKAYA, Russian mathematician, 1922–2004. She worked at Russian Academy of Sciences, St Petersburg, Russia.

[16] James B. SERRIN Jr., American mathematician, born in 1926. He works at University of Minnesota Twin Cities, Minneapolis, MN.

[17] Jacques-Louis LIONS, French mathematician, 1928–2001. He received the Japan Prize in 1991. He held a chair (Analyse mathématique des systèmes et de leur contrôle) at Collège de France, Paris, France. I first had him as a teacher at

In the announcement of the course, I had mentioned that I would start by recalling some classical facts about the way to use functional analysis for solving partial differential equations of continuum mechanics, describing some fine properties of Sobolev[18] spaces which are useful, and studying in detail the spaces adapted to questions about incompressible fluids. I had stated then that the goal of the course was to describe some more recent mathematical models used in oceanography, and show how some of them may be solved, and that, of course, I would point out the known defects of these models.[19] I had mentioned that, for the oceanography part – of which I am no specialist – I would follow a book written by one of my collaborators, Roger LEWANDOWSKI[20] [18], who had learnt about some of these questions from recent lectures of Jacques-Louis LIONS. I mentioned that I was going to distribute notes, from a course on partial differential equations that I had taught a few years before, but as I had not written the part that I had taught on the Stokes equation and the Navier–Stokes equation at the time, I was going to make use of the lecture notes [23] from the graduate course that I had taught at University of Wisconsin, Madison WI, in 1974–1975, where I had added small technical improvements from what I had learnt. Finally, I had mentioned that I would write notes for the parts that I never covered in preceding courses.

I am not good at following plans. I started by reading about oceanography in a book by A. E. GILL[21] [15], and I began the course by describing some of the basic principles that I had learnt there. Then I did follow my plan of discussing questions of functional analysis, but I did not use any of the notes that I had written before. When I felt ready to start describing new models, Roger LEWANDOWSKI visited CARNEGIE[22] MELLON[23] University and gave a talk in the Center for Nonlinear Analysis seminar, and I realized that there were some questions concerning the models and some mathematical techniques which I had not described at all, and I changed my plans. I opted for describing the general techniques for nonlinear partial differential equations that I

École Polytechnique in 1966–1967, and I did research under his direction, until my thesis in 1971.

[18] Sergei L'vovich SOBOLEV, Russian mathematician, 1908–1989. He worked in Novosibirsk, Russia, and there is now a SOBOLEV Institute of Mathematics of the Siberian branch of the Russian Academy of Sciences, Novosibirsk, Russia.

[19] It seems to have become my trade mark among mathematicians, that I do not want to lie about the usefulness of models when some of their defects have already been pointed out. This is obviously the way that any scientist is supposed to behave, but in explaining why I have found myself so isolated and stubborn in maintaining that behavior, I have often invoked a question of religious training.

[20] Roger LEWANDOWSKI, French mathematician, born in 1962. He works at Université de Rennes I, Rennes, France.

[21] Adrian Edmund GILL, Australian-born meteorologist and oceanographer, 1937–1986. He worked in Cambridge, England, UK.

[22] Andrew CARNEGIE, Scottish-born businessman and philanthropist, 1835–1919.

[23] Andrew William MELLON, American financier and philanthropist, 1855–1937.

had developed, *homogenization*, *compensated compactness* and *H-measures*; there are obviously many important situations where they should be useful, and I found it more important to teach them than to analyze in detail some particular models for which I do not feel yet how good they are (which means that I suspect them to be quite wrong). Regularly, I was trying to explain why what I was teaching had some connection with questions about *fluids*.

It goes with my philosophy to *explain the origin of mathematical ideas* when I know about them, and as my ideas are often badly attributed, I like to mention *why and when I had introduced an idea*.

I have also tried to *encourage mathematicians to learn more about continuum mechanics and physics*, listening to the specialists and then trying to put these ideas into a sound mathematical framework. I hope that some of the discussions in these lecture notes will help in this direction.[24]
[This course mentions a few equations from continuum mechanics, and besides the Navier–Stokes equation I shall mention the Maxwell equation, the equation of linearized elasticity, and the wave equation, at least, but I did not always follow the classical notation used in texts of mechanics, writing $a, \mathbf{b}, \mathsf{C}$ for scalars, vectors and tensors, and using the notation $f_{,j}$ for denoting the partial derivative of f with respect to x_j. This course is intended for mathematicians, and even if many results are stated in an informal way, they correspond to theorems whose proofs usually involve functional analysis, and not just differential calculus and linear algebra, which are behind the notation used in mechanics.

It is then important to notice that partial differential equations are not written as pointwise equalities but in the sense of distributions, or more generally in some variational framework and that one deals with elements of function spaces, using operators and various types of convergence. Instead of the notation $\nabla a, \nabla.\mathbf{b}, \nabla \times \mathbf{b}$ used in mechanics, I write $grad\, a, div\, b, curl\, b$ (and I also recall sometimes the framework of differential forms), and I only use $\mathbf{b}$ for a vector-valued function b when the pointwise value is meant, in particular in integrands.

It may seem analogous to the remark known to mathematicians that "the function $f(x)$" is an abuse of language for saying "the function f whose elements in its domain of definition will often be denoted x", but there is something different here. The framework of functional analysis is not just a change of language, because it is crucial for understanding the point of view that I developed in the 1970s for relating what happens at a macroscopic level from the description at a microscopic/mesoscopic level, using convergences of weak type (and not just weak convergences), which is quite a different idea than the game of using ensemble averages, which destroys the physical meaning of the problems considered.]

[24] I have gone further in the critical analysis of many principles of continuum mechanics, which I shall present as a different set of lecture notes, as an introduction to kinetic theory, taught in the Fall of 2001.

Detailed Description of Lectures

a.b refers to definition, lemma or theorem # b in lecture # a, while (a.b) refers to equation # b in lecture # a.

Lecture 1, Basic physical laws and units: The hypothesis of incompressibility and the speed of sound in water; salinity; units in the metric system; oceanography/meteorology; energy received from the Sun: the solar constant S; black-body radiation, Planck's law, surface temperature of the Sun; absorption, albedo, the greenhouse effect; convection of water induced by gravity and temperature, and salinity; how a greenhouse functions.
Lecture 2, Radiation balance of atmosphere: The observed percentages of energy in the radiation balance of the atmosphere; absorption and emission are frequency-dependent effects; the greenhouse with p layers (2.1)–(2.7); thermodynamics of air and water: lapse rate, relative humidity, latent heat; the Inter-Tropical Convergence Zone (ITCZ), the trade winds, cyclones and anti-cyclones.
Lecture 3, Conservations in ocean and atmosphere: The differences between atmosphere and ocean concerning heat storage; conservation of angular momentum, the trade winds, east–west dominant wings; conservation of salt; Eulerian and Lagrangian points of view; conservation of mass (3.1)–(3.3).
Lecture 4, Sobolev spaces I: Sobolev spaces $W^{1,p}(\Omega)$ (4.1)–(4.2); weak derivatives, theory of distributions; notation H^s for $p = 2$ and $\mathcal{H}$ for Hardy spaces; functions of $W^{1,p}(\Omega)$ have a trace on $\partial\Omega$ if it is smooth; integration by parts in $W^{1,1}(\Omega)$ (4.3); results from ordinary differential equations (4.4)–(4.8); conservation of mass (4.9)–(4.12); regularity of solutions of the Navier–Stokes and Euler equations, Riesz operators and singular integrals, Zygmund space, BMO, $\mathcal{H}^1$.
Lecture 5, Particles and continuum mechanics: Particles and continuum mechanics, distances between molecules; homogenization, microscopic/mesoscopic/macroscopic scales; "real" particles versus macroscopic particles as tools from numerical analysis; Radon measures (5.1), distributions (5.2)–(5.4);

momentum and conservation of mass (5.5)–(5.6); the homogenization problem related to oscillations in the velocity field.
Lecture 6, Conservation of mass and momentum: Euler equation (6.1); priority of Navier over Stokes and of Stokes over Riemann, Rankine and Hugoniot; similarity of the stationary Stokes equation and stationary linearized elasticity; kinetic theory, free transport equation and conservation of mass (6.2)–(6.4); transport equation with Lorentz force (6.5); Boltzmann's equation (6.6)–(6.7); Cauchy stress in kinetic theory (6.8); conservation of momentum (6.10); pressure on the boundary resulting from reflection of particles.
Lecture 7, Conservation of energy: Internal energy in kinetic theory (7.1); relation between internal energy and Cauchy stress in kinetic theory (7.2); heat flux in kinetic theory (7.3); conservation of energy (7.4); various origins of the internal energy; variation of thermodynamic entropy, H-theorem (7.5)–(7.6); local Maxwellian distribution (7.7); the parametrization of allowed collisions (7.8)–(7.9); the form of interaction term $Q(f,f)$ in Boltzmann's equation (7.10); the proof of (7.5): (7.11); letting the mean free path tend to 0; irreversibility, nonnegative character of solutions of Boltzmann's equation (7.12)–(7.13).
Lecture 8, One-dimensional wave equation: Longitudinal, transversal waves; approximating the longitudinal vibration of a string by small masses connected with springs (8.1)–(8.2); the limiting 1-dimensional wave equation (8.3)–(8.5); different scalings of string constants; time periodic solutions; linearization for the increase in length in 1-dimensional transversal waves and 2- or 3-dimensional problems; the linearized elasticity system (8.6)–(8.11); Cauchy's introduction of the stress tensor, by looking at the equilibrium of a small tetrahedron.
Lecture 9, Nonlinear effects, shocks: Beware of linearization; nonlinear string equation (9.1); Poisson's study of barotropic gas dynamics with $p = C\,\varrho^\gamma$ (9.2); what led Stokes to discover "Rankine–Hugoniot" conditions; Burgers's equation (9.3)–(9.5); characteristic curves and apparition of discontinuities (9.6)–(9.7); equations in the sense of distributions imply jump conditions (9.8)–(9.9); a two-parameter family of weak solutions for Burgers's equation with 0 initial datum (9.10); Lax's condition and Oleinik's condition for selecting admissible discontinuities; Hopf's derivation of Oleinik's condition using "entropies" (9.11)–(9.13), Lax's extension to systems; the equation for entropies of system (9.14) describing the nonlinear string equation (9.15)–(9.17); transonic flows.
Lecture 10, Sobolev spaces II: Description of functional spaces for the study of the Stokes and Navier–Stokes equations, boundedness of Ω, smoothness of $\partial\Omega$; $H^1(\Omega)$ (10.1), characteristic length, $H^1_0(\Omega)$; Poincaré's inequality (10.2); scaling, Poincaré's inequality does not hold for open sets containing arbitrary large balls (10.3)–(10.4); 10.1: Poincaré's inequality holds if Ω is included in a bounded strip (10.5), if $meas\,\Omega < \infty$ (10.11)–(10.12); Schwartz's convention for the Fourier transform (10.6), its action on derivation and multiplication (10.7); Plancherel's formula (10.8); Schwartz's extension of the Fourier transform to temperate distributions (10.9); the Fourier transform is an isometry

on $L^2(R^N)$ (10.10); a sufficient condition for having Poincaré's inequality; the strain–stress constitutive relation in isotropic linearized elasticity (10.13).
Lecture 11, Linearized elasticity: Stationary linearized elasticity for isotropic materials (11.1)–(11.3); 11.1: Korn's inequality on $H_0^1(\Omega; R^N)$ (11.4), a proof using the Fourier transform (11.5), a proof by integration by parts (11.6)–(11.8); 11.2: Lax–Milgram lemma (11.9)–(11.10); variational formulation and approximation; the complex-valued case of the Lax–Milgram lemma; 11.3: a variant of the Lax–Milgram lemma (11.11); description of the plan for letting $\lambda \to +\infty$.
Lecture 12, Ellipticity conditions: Very strong ellipticity condition (12.1), the isotropic case; strong ellipticity condition (12.2) for stationary linearized elasticity, the isotropic case, the constant coefficients case with Dirichlet condition; the abstract framework for letting $\lambda \to +\infty$ in linearized elasticity (12.3)–(12.4), bounds for u^λ (12.5), variational form of the limit problem (12.6)–(12.7), strong convergence of u^λ (12.8)–(12.9); Lagrange multiplier; definition and characterization of $H^{-1}(\Omega)$ the dual of $H_0^1(\Omega)$ (12.10)–(12.11); equations satisfied by u^λ and its limit u^∞ (12.12)–(12.14); De Rham's theorem and interpretation of (12.14); $grad\, S \in H^{-1}(\Omega; R^N)$ implies $S \in L^2(\Omega)$ if $\partial\Omega$ is smooth.
Lecture 13, Sobolev spaces III: $X(\Omega)$ (13.1); relation with Korn's inequality (13.2); 13.1: existence of the "pressure", and 13.2: existence of $u \in H_0^1(\Omega; R^N), div\, u = g$ whenever $\int_\Omega g\, dx = 0$, are equivalent if $\partial\Omega$ is smooth; proof based on regularity for a degenerate elliptic problem; 13.3: the equivalence lemma; applications of the equivalence lemma; 13.4: $X(R^N) = L^2(R^N)$ using the Fourier transform.
Lecture 14, Sobolev spaces IV: Approximation methods in $W^{1,p}(\Omega)$; truncation; properties of convolution in R^N (14.1)–(14.2); regularization by convolution (14.3); commutation of convolution and derivation (14.4), $C^\infty(R^N)$ is dense in $W^{1,p}(R^N)$; support of convolution product (14.5)–(14.6), $C^\infty(\overline{R_+^N})$ is dense in $W^{1,p}(R_+^N)$ for $\Omega = R^N$; localization, partition of unity, $C^\infty(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$ when Ω is bounded and $\partial\Omega$ is locally a continuous graph; extension from $W^{m,p}(R_+^N)$ to $W^{m,p}(R^N)$ (14.7)–(14.9); counter-example to the extension from $H^1(\Omega)$ to $H^1(R^2)$ for a plane domain with a cusp.
Lecture 15, Sobolev spaces V: $X(\Omega)$ is a local space; $C_c^\infty(\overline{R_+^N})$ is dense in $X(R_+^N)$; extension from $X(R_+^N)$ to $X(R^N)$ by transposition and construction of a restriction (15.1)–(15.3); the importance of regularity of $\partial\Omega$ for having $X(\Omega) = L^2(\Omega)$; 15.1: if $meas(\Omega) < \infty$, the embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ is compact, by the Fourier transform; application to the convergence of $-\lambda\, div\, u^\lambda$ in $L^2(\Omega)$ to the "pressure", by the equivalence lemma.
Lecture 16, Sobolev embedding theorem: Differences between linearized elasticity and the Stokes equation for the evolution problems; variable viscosity, Poiseuille flows; stationary Navier–Stokes equation (16.1); 16.1: Sobolev embedding theorem, the original method of Sobolev and improvements using interpolation spaces, an inequality of Ladyzhenskaya (16.2) and a method of

Gagliardo and of Nirenberg (16.3)–(16.5); solving (16.1) as fixed point for Φ (16.6), estimates for Φ giving existence and uniqueness of a solution for small data and $N \leq 4$ (16.7)–(16.12), by the Banach fixed point theorem; solving (16.1) as fixed point for Ψ (16.13), estimates for Ψ (16.14)–(16.20); 16.2: existence of a fixed point for a contraction of a closed bounded nonempty convex set in a Hilbert space, monotone operators.
Lecture 17, Fixed point theorems: Existence of a solution of (16.1) for large data by the Schauder fixed point theorem for $N \leq 3$, by the Tykhonov fixed point theorem for $N = 4$; Faedo–Ritz–Galerkin method; existence of Faedo–Ritz–Galerkin approximations (17.1) by the Brouwer fixed point method applied to approximations Ψ_m (17.2), existence for large data for $N \leq 4$ by extraction of weakly converging subsequence and a compactness argument, valid for $N > 4$ in larger functional spaces; properties of the Brouwer topological degree; 17.1: nonexistence of tangent nonvanishing vector fields on S^{2N}; 17.2: nonexistence of a continuous retraction of a bounded open set of R^N onto its boundary; 17.3: Brouwer fixed point theorem.
Lecture 18, Brouwer's topological degree: $J_\varphi(u)$ (18.1); 18.1: the derivative of $J_\varphi(u)$ in the direction v is an integral on $\partial\Omega$ (18.2)–(18.3), vanishing if v vanish on $\partial\Omega$; 18.2: invariance by homotopy, $J_\varphi(u) = J_\varphi(w)$ if there is a homotopy from u to w avoiding $supp(\varphi)$ on $\partial\Omega$; 18.3: $J_\varphi(u)$ can be defined for $u \in C(\overline{\Omega}; R^N)$ avoiding $supp(\varphi)$ on $\partial\Omega$; 18.4: if $J_\varphi(u) \neq 0$ there exists $\mathbf{x} \in \Omega$ such that $\mathbf{u}(\mathbf{x}) \in supp(\varphi)$; proof of 18.1: (18.4)–(18.7); 18.5: definition of degree $deg(u; \Omega, \mathbf{p})$; 18.6: formula for degree if $\mathbf{u}(\mathbf{z}) = \mathbf{p}$ has a finite number of solutions where ∇u is invertible (18.8); Sard's lemma.
Lecture 19, Time-dependent solutions I: Spaces V, H for the Stokes or Navier–Stokes equations (19.1)–(19.2); semi-group theory; abstract ellipticity for $A \in \mathcal{L}(V, V')$ (19.3); 19.1: $u' + A\,u = f \in L^1(0,T;H) + L^2(0,T;V')$, $u(0) = u_0 \in H$ (19.4)–(19.5), by Faedo–Ritz–Galerkin (19.6); 19.2: properties of $W^{1,1}(0,T)$ and Gronwall's inequality; estimates for (19.6): (19.7)–(19.16); a variant of Gronwall's inequality (19.17)–(19.19), giving estimate (19.20).
Lecture 20, Time-dependent solutions II: Taking the limit in (19.6), (20.1)–(20.3), giving existence in 19.1; an identity for proving uniqueness in 19.1, (20.4); spaces $W_1(0,T)$ and $W(0,T)$ (20.5)–(20.8); properties of $W_1(0,T)$, for proving (20.4); problem with time derivative in Faedo–Ritz–Galerkin, and special choice for a basis; regularization effect when the initial datum is not in the right space; backward uniqueness in the case $A^T = A$, Agmon–Nirenberg result of log-convexity for $|u(t)|$.
Lecture 21, Time-dependent solutions III: Problem in the definition of H in (19.2); problem with the "pressure" in the nonstationary Stokes equation (21.1)–(21.7); 21.1: regularity in space when $A^T = A$, $u_0 \in V, f \in L^2(0,T;H)$, regularizing effect for $u_0 \in H, \sqrt{t}\,f \in L^2(0,T;H)$; problem of identifying H' with H; estimate for the "pressure" in the case $\Omega = R^N$ (21.8)–(21.11); avoiding cutting the transport operator into two terms (21.12)–(21.14); the nonlinear term (21.15) and its estimate in dimension 2, 3, 4 (21.16)–(21.17).

Lecture 22, Uniqueness in 2 dimensions: Cutting the transport term into two terms works for $N = 2$; 21.1: uniqueness for the abstract Navier–Stokes equation for $N = 2$ (21.1)–(21.6); a quasilinear diffusion equation (21.7), with the Artola uniqueness result (21.8)–(21.11).
Lecture 23, Traces: $H(div; \Omega)$ (23.1); space is local, $C^\infty(\overline{\Omega}; R^N)$ dense if $\partial\Omega$ smooth; formula defining the normal trace $u.\nu$ (23.2), in dual of traces of $H^1(\Omega)$ (23.3); interpretation in terms of differential forms, $H(curl; \Omega)$ (23.4); $H^s(R^N)$ (23.5); for $s > 1/2$, restriction on $x_N = 0$ is defined on $H^s(R^N)$, and the trace space is $H^{s-(1/2)}(R^{N-1})$ (23.6)–(23.10); 23.1: orthogonal of H in $L^2(\Omega; R^N)$ is the space $\{grad(p) \mid p \in H^1(\Omega)\}$, if injection of $H^1(\Omega)$ into $L^2(\Omega)$ is compact; 23.2: if $meas\,\Omega < \infty$ and $X(\Omega) = L^2(\Omega)$ then V is dense in H; discussion of $X(\Omega) = L^2(\Omega)$ if $\partial\Omega$ is smooth, and how to change the definitions of the spaces if the boundary is not smooth enough; Faedo–Ritz–Galerkin method for existence of Navier–Stokes equation for $N = 3$ (23.11)–(23.12); singular solutions of the stationary Stokes equation in corners (23.13)–(23.18).
Lecture 24, Using compactness: 24.1: J.-L. Lions's lemma (24.1); 24.2: u_n bounded in $L^p(0,T;E_1)$ and convergent in $L^p(0,T;E_3)$ imply u_n convergent in $L^p(0,T;E_2)$ if injection of E_1 into E_2 is compact (24.2); 24.3: u_n bounded in $L^{p_1}(0,T;E_1)$ and convergent in $L^{p_3}(0,T;E_3)$ gives u_n convergent in $L^{p_2}(0,T;E_2)$ if interpolation inequality holds; hypothesis of reflexivity; 24.4: u_n bounded in $L^p(0,T;E)$ and $||\tau_h u_n - u_n||_{L^p(0,T;E)} \le M\,|h|^\eta$ imply u_n bounded in $L^q(0,T;E)$; 24.5: u_n bounded in $L^p(0,T;E_1)$ and $||\tau_h u_n - u_n||_{L^p(0,T;E_3)} \le M\,|h|^\eta$ imply u_n compact in $L^p(0,T;E_2)$ if injection of E_1 into E_2 is compact; application to extracting subsequences from Faedo–Ritz–Galerkin approximation with special basis for the Navier–Stokes equation and $N \le 3$.
Lecture 25, Existence of smooth solutions: 25.1: If $N = 2$ and Ω smooth enough, $u_0 \in V$ and $f \in L^2((0,T)\times\Omega; R^2)$ then regularity of the linear case holds (25.1)–(25.2); can one improve bounds using interpolation inequalities; 25.2: if $N = 3$ and Ω smooth enough, $u_0 \in V$ and $f \in L^2((0,T)\times\Omega; R^3)$ then there exists $T_c \in (0,T]$ and a solution with the regularity of the linear case for $t \in (0,T_c)$ (25.3)–(25.4); 25.3: if $N = 3$ and Ω smooth enough, $|u_0|\,||u_0||$ small and $f = 0$ then a global solution with the regularity of the linear case exists for $t \in (0,\infty)$ (25.5)–(25.7); the case $f \neq 0$ (25.8); extending an idea of Foias for showing $u \in L^1(0,T;L^\infty(\Omega;R^3))$ for $N = 3$ (25.9)–(25.12).
Lecture 26, Semilinear models: Reynolds number, scaling of norms, the problems that norms give global information and not local information; a different approach shown on models of kinetic theory, the 2-dimensional Maxwell model (26.1), Broadwell model (26.2); using functional spaces with physical meaning; a special class of semilinear models (26.3)–(26.4) and why I had introduced it; 26.1: spaces $V_c \subset W_c$ and L^1 estimate in (x,t) for $u\,v$ (26.5)–(26.7); extension of the idea, compensated integrability.
Lecture 27, Size of singular sets: Leray's self-similar solutions (27.1); the question of estimating the Hausdorff dimension of singular sets; a bound for the

1/2 Hausdorff dimension in t (27.2); different scaling in $(\mathbf{x}, t)$ and the equation for "pressure" (27.3); maximal functions (27.4), Hedberg's program of proving local inequalities using maximal functions (27.5), application to pointwise estimates for the heat equation (27.6)–(27.10).

Lecture 28, Local estimates, compensated integrability: Hedberg's truncation method, a proof of F.-C. Liu's inequality using Hedberg's approach (28.1)–(28.2), a Hedberg type version of the Gagliardo–Nirenberg inequality (28.3); a result of compensated integrability improving Wente by estimates based on interpolation and Lorentz spaces.

Lecture 29, Coriolis force: Equations in a moving frame and Coriolis force (29.1)–(29.3); analogy, Lorentz force, incompressible fluid motion, nonlinearity as $\mathbf{u} \times curl(-\mathbf{u}) + grad(|\mathbf{u}|^2/2)$ (29.4)–(29.8), conservation of helicity.

Lecture 30, Equation for the vorticity: Equation for vorticity, for $N = 2$ and for $N = 3$ (30.1)–(30.6).

Lecture 31, Boundary conditions in linearized elasticity: Other boundary conditions for linearized elasticity, Neumann condition (31.1) and compatibility conditions (31.2)–(31.3); studying linearized rigid displacements (31.4); other type of boundary conditions; traction at the boundary for a Newtonian fluid (31.5)–(31.6).

Lecture 32, Turbulence, homogenization: Microstructures in turbulent flows; the defect of probabilistic postulates; homogenization.

Lecture 33, G-convergence and H-convergence: Weak convergence, linear partial differential equations in theory of distributions; conservation of mass using differential forms; G-convergence and H-convergence; exterior calculus, differential forms, exterior derivative, Poincaré lemma; weak convergence as a way to relate mesoscopic and macroscopic levels, analogy between proofs in H-convergence and the way some physical quantities are measured and other physical quantities are identified; 33.1: div-curl lemma, its relation with differential forms.

Lecture 34, One-dimensional homogenization, Young measures: 1-dimensional homogenization by div-curl lemma; the G-convergence and H-convergence approaches; effective coefficients cannot be computed in terms of Young measures in dimension $N \geq 2$, physicists' formulas are approximations; importance of both balance equations and constitutive relations; 34.1: Young measures.

Lecture 35, Nonlocal effects I: Turbulence as an homogenization problem for a first order transport operator (35.1); memory effects appearing by homogenization; a model problem with a memory effect in its effective equation (35.2)–(35.3), proof by the Laplace transform (35.4)–(35.9); irreversibility without probabilistic framework; a transport problem with a nonlocal effect in (x, t) in its effective equation (35.10)–(35.15).

Lecture 36, Nonlocal effects II: Frequency-dependent coefficients in Maxwell's equation (36.1), principle of causality, pseudo-differential operators; the model problem with time dependent coefficients (36.1)–(36.8), by a perturbation expansion approach; "analogies" with Feynman diagrams and Padé approximants.

Lecture 37, A model problem: A model problem with a term $\mathbf{u} \times curl(\mathbf{v}_n)$ added to the stationary Stokes equation (37.1)–(37.2), the derivation of the effective equation (37.3)–(37.14), by methods from H-convergence; an effective term corresponding to a dissipation quadratic in $\mathbf{u}$ and not in $grad\,\mathbf{u}$, which can be computed with H-measures.

Lecture 38, Compensated compactness I: The time dependent analog requires a variant of H-measures; 38.1: the quadratic theorem of compensated compactness (38.1)–(38.4); chronology of discoveries; correction for $U \otimes U$ written as the computation of a convex hull, a formula simplified by introduction of H-measures.

Lecture 39, Compensated compactness II: Constitutive relations (39.1), balance equations (39.2), question about how to treat nonlinear elasticity (39.3); H-measures can handle variable coefficients; how compensated compactness constrains Young measures (39.4); examples: compactness, convexity, monotonicity, Maxwell's equation; proof of necessary conditions.

Lecture 40, Differential forms: Maxwell's equation expressed with differential forms (40.1)–(40.6); 40.1: generalization of div-curl lemma for p-forms and q-forms; generalizations to Jacobians, special case of exact forms (40.7); 40.2: one cannot use the weak topology in the general div-curl lemma; other necessary conditions; how helicity appears in the framework of differential forms, analogy between Lorentz force and the equations for fluid flows.

Lecture 41, The compensated compactness method: 41.1: case when the characteristic set is the zero set of a nondegenerate quadratic form; the question of making the list of interesting quantities in nonlinear elasticity (41.1); wave equation (41.2), conservation of energy (41.3), where the energy goes, equipartition of energy; use of entropies for Burgers's equation for passing to the limit for weakly converging sequences (41.4)–(41.12), entropy condition (41.13) and Murat's lemma.

Lecture 42, H-measures and variants: Wigner transform, avoiding using one characteristic length, the hints for H-measures; definitions for H-measures (42.1)–(42.4); constructing the right "pseudo-differential" calculus (42.5)–(42.12); localization principle (42.13)–(42.14); small-amplitude homogenization (42.15)–(42.18); propagation equations for H-measures (42.19)–(42.27); the variant with one characteristic length, semi-classical measures of P. Gérard (42.28).

Biographical data: Basic biographical information for people whose name is associated with something mentioned in the lecture notes.

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