Roger Temam

# Infinite-Dimensional Dynamical Systems in Mechanics and Physics

With 13 Illustrations



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