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Variational Methods

Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems

Second, Revised and Substantially Expanded Edition

With 16 Figures

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Preface to the Second Edition

During the short period of five years that have elapsed since the publication of the first edition a number of interesting mathematical developments have taken place and important results have been obtained that relate to the theme of this book.

First of all, as predicted in the Preface to the first edition, Morse theory, indeed, has gone through a dramatic change, influenced by the work by Andreas Floer on Hamiltonian systems and in particular, on the Arnold conjecture. There are now also excellent accounts of these developments and their ramifications; see, in particular, the monograph by Matthias Schwarz [1]. The book by Hofer-Zehnder [2] on Symplectic Geometry shows that variational methods and, in particular, Floer theory have applications that range far beyond the classical area of analysis.

Second, as a consequence of an observation by Stefan Müller [1] which prompted the seminal work of Coifman-Lions-Meyer-Semmes [1], Hardy spaces and the space BMO are now playing a very important role in weak convergence results, in particular, when dealing with problems that exhibit a special (determinant) structure. A brief discussion of these results and some model applications can be found in Section I.3.

Moreover, variational problems depending on some real parameter in certain cases have been shown to admit rather surprising a-priori bounds on critical points, with numerous applications. Some examples will be given in Chapters I.7 and II.9.

Other developments include the discovery of Hamiltonian systems with no periodic orbits on some given energy hypersurface, due to Ginzburg and Herman, and the discovery, by Chang-Ding-Ye, of finite time blow-up for the evolution problem for harmonic maps of surfaces, thus completing the results in Sections II.8, II.9 and III.6, respectively.

A beautiful recent result of Ye concerns a new proof of the Yamabe theorem in the case of a locally conformally flat manifold. This proof is presented in detail in Section III.4 of this new edition.

In view of their numerous and wide-ranging applications, interest in variational methods is very strong and growing. Out of the large number of recent publications in the general field of the calculus of variations and its applications some 50 new references have been added that directly relate to one of the themes in this monograph.

Owing to the very favorable response with which the first edition of this book was received by the mathematical community, the publisher has suggested that a second edition be published in the Ergebnisse series. It is a pleasure to thank all the many mathematicians, colleagues, and friends who have commented on the first edition. Their enthusiasm has been highly inspiring. Moreover, I would like to thank, in particular, Matts Essen, Martin Flucher and Helmut Hofer for helpful suggestions in preparing this new edition.

All additions and changes to the first edition were carefully implemented by Suzanne Kronenberg, using the Springer TeX-Macros package, and I gratefully acknowledge her help.

Zürich, Juni 1996 Michael Struwe

Preface to the First Edition

It would be hopeless to attempt to give a complete account of the history of the calculus of variations. The interest of Greek philosophers in isoperimetric problems underscores the importance of "optimal form" already in ancient cultures; see Hildebrandt-Tromba [1] for a beautiful treatise of this subject. While variational problems thus are part of our classical cultural heritage, the first modern treatment of a variational problem is attributed to Fermat, see Goldstine [1; p.1]. Postulating that light follows a path of least possible time, in 1662 Fermat was able to derive the laws of refraction, thereby using methods which may already be termed analytic.

With the development of the Calculus by Newton and Leibniz, the basis was laid for a more systematic development of the calculus of variations. The brothers Johann and Jakob Bernoulli and Johann's student Leonhard Euler, all from the city of Basel in Switzerland, were to become the "founding fathers" (Hildebrandt-Tromba [1; p.21]) of this new discipline. In 1743 Euler [1] submitted "A method for finding curves enjoying certain maximum or minimum properties", published 1744, the first textbook on the calculus of variations. In an appendix to this book Euler [1; Appendix II, p. 298] expresses his belief that "every effect in nature follows a maximum or minimum rule" (see also Goldstine [1; p. 106]), a credo in the universality of the calculus of variations as a tool. The same conviction also shines through Maupertuis' [1] work on the famous "least action principle", also published in 1744. (In retrospect, however, it seems that Euler was the first to observe this important principle. See for instance Goldstine [1; p. 67 f. and p. 101 ff.] for a more detailed historical account.) Euler's book was a great source of inspiration for generations of mathematicians following.

Major contributions were made by Lagrange, Legendre, Jacobi, Clebsch, Mayer, and Hamilton to whom we owe what we now call "Euler-Lagrange equations", the "Jacobi differential equation" for a family of extremals, or "Hamilton-Jacobi theory".

The use of variational methods was not at all limited to 1-dimensional problems in the mechanics of mass-points. In the 19^{th} century variational methods also were employed for instance to determine the distribution of an electrical charge on the surface of a conductor from the requirement that the energy of the associated electrical field be minimal ("Dirichlet's principle"; see Dirichlet [1] or Gauss [1]) or were used in the construction of analytic functions (Riemann [1]).

However, none of these applications was carried out with complete rigor. Often the model was confused with the phenomenon that it was supposed to describe and the fact (?) that for instance in nature there always exists a equilibrium distribution for an electrical charge on a conducting surface was taken as sufficient evidence for the corresponding mathematical problem to have a solution. A typical reasoning reads as follows:

"In any event therefore the integral will be non-negative and hence there must exist a distribution (of charge) for which this integral assumes its minimum value," (Gauss [1; p.232], translation by the author).

However, towards the end of the 19^{th} century progress in abstraction and a better understanding of the foundations of the calculus opened such arguments to criticism. Soon enough, Weierstrass [1; pp. 52–54] found an example of a variational problem that did not admit a minimum solution. Weierstrass challenged his colleagues to find a continuously differentiable function $u\colon [-1,1] \to \mathbb{R}$ minimizing the integral

$$I(u) = \int_{-1}^{1} \left| x \frac{d}{dx} u \right|^2 \, dx$$

subject (for instance) to the boundary conditions $u(\pm 1) = \pm 1$. Choosing

$$u_\varepsilon(x) = \frac{\arctan(\frac{x}{\varepsilon})}{\arctan(\frac{1}{\varepsilon})}, \ \varepsilon > 0,$$

as a family of comparison functions, Weierstrass was able to show that the infinium of I in the above class was 0; however, the value 0 is not attained. (See also Goldstine [1; p. 371 f.].) Weierstrass' critique of Dirichlet's principle precipitated the calculus of variations into a Grundlagenkrise comparable to the crisis in set theory and logic after Russel's discovery of antinomies in Cantor's set theory or Gödel's incompleteness proof.

However, through the combined efforts of several mathematicians who did not want to give up the wonderful tool that Dirichlet's principle had been – including Weierstrass, Arzéla, Fréchet, Hilbert, and Lebesgue – the calculus of variations was revalidated and emerged from its crisis with new strength and vigor.

Hilbert's speech at the centennial assembly of the International Congress 1900 in Paris, where he proposed his famous 20 problems – two of which devoted to questions related to the calculus of variatons – marks this newly found confidence.

In fact, following Hilbert's [1] and Lebesgue's [1] solution of the Dirichlet problem, a development began which within a few decades brought tremendous success, highlighted by the 1929 theorem of Ljusternik and Schnirelman [1] on the existence of three distinct prime closed geodesics on any compact surface of genus zero, or the 1930/31 solution of Plateau's problem by Douglas [1], [2] and Radò [1].

The Ljusternik-Schnirelman result (and a previous result by Birkhoff [1], proving the existence of one closed geodesic on a surface of genus 0) also marks the beginning of global analyis. This goes beyond Dirichlet's principle as we no longer consider only minimizers (or maximizers) of variational integrals, but instead look at all their critical points. The work of Ljusternik and Schnirelman revealed that much of the complexity of a function space is invariably reflected in the set of critical points of any variational integral defined on it, an idea whose importance for the further development of mathematics can hardly be overestimated, whose implications even today may only be conjectured, and whose applications seem to be virtually unlimited. Later, Ljusternik and Schnirelman [2] laid down the foundations of their method in a general theory. In honor of their pioneering effort any method which seeks to draw information concerning the number of critical points of a functional from topological data today often is referred to as Ljusternik-Schnirelman theory.

Around the time of Ljusternik and Schnirelman's work, another – equally important – approach towards a global theory of critical points was pursued by Marston Morse [2]. Morse's work also reveals a deep relation between the topology of a space and the number and types of critical points of any function defined on it. In particular, this led to the discovery of unstable minimal surfaces through the work of Morse-Tompkins [1], [2] and Shiffman [1], [2]. Somewhat reshaped and clarified, in the 50's Morse theory was highly successful in topology (see Milnor [1] and Smale [1]). After Palais [1], [2] and Smale [2] in the 60's succeeded in generalizing Milnor's constructions to infinite-dimensional Hilbert manifolds – see also Rothe [1] for some early work in this regard – Morse theory finally was recognized as a useful (and usable) instrument also for dealing with partial differential equations.

However, applications of Morse theory seemed somewhat limited in view of prohibitive regularity and non-degeneracy conditions to be met in a variational problem, conditions which – by the way – were absent in Morse's original work. Today, inspired by the deep work of Conley [1], Morse theory seems to be turning back to its origins again. In fact, a Morse-Conley theory is emerging which one day may provide a tool as universal as Ljusternik-Schnirelman theory and still offer an even better resolution of the relation between the critical set of a functional and topological properties of its domain. However, in spite of encouraging results, for instance by Benci [4], Conley-Zehnder [1], Jost-Struwe [1], Rybakowski [1], [2], Rybakowski-Zehnder [1], Salamon [1], and – in particular – Floer [1], a general theory of this kind does not yet exist.

In these notes we want to give an overview of the state of the art in some areas of the calculus of variations. Chapter I deals with the classical direct methods and some of their recent extensions. In Chapters II and III we discuss minimax methods, that is, Ljusternik-Schnirelman theory, with an emphasis on some limiting cases in the last chapter, leaving aside the issue of Morse theory whose face is currently changing all too rapidly.

Examples and applications are given to semilinear elliptic partial differential equations and systems, Hamiltonian systems, nonlinear wave equations, and problems related to harmonic maps of Riemannian manifolds or surfaces of prescribed mean curvature. Although our selection is of course biased by the interests of the author, an effort has been made to achieve a good balance between different areas of current research. Most of the results are known;

some of the proofs have been reworked and simplified. Attributions are made to the best of the author's knowledge. No attempt has been made to give an exhaustive account of the field or a complete survey of the literature.

General references for related material are Berger-Berger [1], Berger [1], Chow-Hale [1], Eells [1], Nirenberg [1], Rabinowitz [11], Schwartz [2], Zeidler [1]; in particular, we recommend the recent books by Ekeland [2] and Mawhin-Willem [1] on variational methods with a focus on Hamiltonian systems and the forthcoming works of Chang [7] and Giaquinta-Hildebrandt. Besides, we mention the classical textbooks by Krasnoselskii [1] (see also Krasnoselskii-Zabraiko [1]), Ljusternik-Schnirelman [2], Morse [2], and Vainberg [1]. As for applications to Hamiltonian systems and nonlinear variational problems, the interested reader may also find additional references on a special topic in these fields in the short surveys by Ambrosetti [2], Rabinowitz [9], or Zehnder [1].

The material covered in these notes is designed for advanced graduate or Ph.D. students or anyone who wishes to acquaint himself with variational methods and possesses a working knowledge of linear functional analysis and linear partial differential equations. Being familiar with the definitions and basic properties of Sobolev spaces as provided for instance in the book by Gilbarg-Trudinger [1] is recommended. However, some of these prerequisites can also be found in the appendix.

In preparing this manuscript I have received help and encouragement from a number of friends and colleagues. In particular, I wish to thank Proff. Herbert Amann and Hans-Wilhelm Alt for helpful comments concerning the first two sections of Chapter I. Likewise, I am indebted to Prof. Jürgen Moser for useful suggestions concerning Section I.4 and to Proff. Helmut Hofer and Eduard Zehnder for advice on Sections I.6, II.5, and II.8, concerning Hamiltonian systems.

Moreover, I am grateful to Gabi Hitz, Peter Bamert, Jochen Denzler, Martin Flucher, Frank Josellis, Thomas Kerler, Malte Schünemann, Miguel Sofer, Jean-Paul Theubet, and Thomas Wurms for going through a set of preliminary notes for this manuscript with me in a seminar at ETH Zürich during the winter term of 1988/89. The present text certainly has profited a great deal from their careful study and criticism.

Special thanks I also owe to Kai Jenni for the wonderful typesetting of this manuscript with the TEX text processing system.

I dedicate this book to my wife Anne.

Zürich, January 1990 Michael Struwe

Contents

Glossary of Notations

V, V^*	generic Banach space with dual V^*
$\lVert \cdot \rVert$	norm in V
$\lVert \cdot \rVert_*$	induced norm in V^*, often also denoted $\lVert \cdot \rVert$
$\langle \cdot, \cdot \rangle : V \times V^* \to \mathbb{R}$	dual pairing, occasionally also used to denote scalar product in $\mathbb{R}^n$
E	generic energy functional
DE	Fréchet derivative
$\mathrm{Dom}(E)$	domain of E
$\langle v, DE(u) \rangle = DE(u)v = D_v E(u)$	directional derivative of E at u in direction v
$L^p(\Omega; \mathbb{R}^n)$	space of Lebesgue-measurable functions $u: \Omega \to \mathbb{R}^n$ with finite L^p-norm $$\lVert u \rVert_{L^p} = \Big(\int_\Omega \lvert u \rvert^p \, dx \Big)^{1/p}, \ 1 \le p < \infty$$
$L^\infty(\Omega; \mathbb{R}^n)$	space of Lebesgue-measurable and essentially bounded functions $u: \Omega \to \mathbb{R}^n$ with norm $$\lVert u \rVert_{L^\infty} = \operatorname*{ess\,sup}_{x \in \Omega} \lvert u(x) \rvert .$$
$H^{m,p}(\Omega; \mathbb{R}^n)$	Sobolev space of functions $u \in L^p(\Omega; \mathbb{R}^n)$ with $\lvert \nabla^k u \rvert \in L^p(\Omega)$ for all $k \in \mathbb{N}_0^n, \lvert k \rvert \le m$, with norm $\lVert u \rVert_{H^{m,p}} = \sum_{0 \le \lvert k \rvert \le m} \lVert \nabla^k u \rVert_{L^p}$.
$H_0^{m,p}(\Omega; \mathbb{R}^n)$	completion of $C_0^\infty(\Omega; \mathbb{R}^n)$ in the norm $\lVert \cdot \rVert_{H^{m,p}}$; if Ω is bounded an equivalent norm is given by $\lVert u \rVert_{H_0^{m,p}} = \sum_{\lvert k \rvert = m} \lVert \nabla^k u \rVert_{L^p}$.
$H^{-m,q}(\Omega; \mathbb{R}^n)$	dual of $H_0^{m,p}(\Omega; \mathbb{R}^n)$, where $\frac{1}{p} = \frac{1}{q} = 1$; q is omitted, if $p = q = 2$.
$D^{m,p}(\Omega; \mathbb{R}^n)$	completion of $C_0^\infty(\Omega; \mathbb{R}^n)$ in the norm $\lVert u \rVert_{D^{m,p}} = \sum_{\lvert k \rvert = m} \lVert \nabla^k u \rVert_{L^p}$.

$C^{m,\alpha}(\Omega; \mathbb{R}^n)$	space of m times continuously differentiable functions $u\colon \Omega \to \mathbb{R}^n$ whose m-th order derivatives are Hölder continuous with exponent $0 \le \alpha \le 1$
$C_0^\infty(\Omega; \mathbb{R}^n)$	space of smooth functions $u\colon \Omega \to \mathbb{R}^n$ with compact support in Ω.
$\mathrm{supp}(u) = \overline{\{x \in \Omega\ ;\ u(x) \neq 0\}}$	support of a function $u\colon \Omega \to \mathbb{R}^n$.
$\Omega' \subset\subset \Omega$	the closure of Ω' is compact and contained in Ω
$\llcorner$	restriction of a measure
$\mathcal{L}^n$	Lebesgue measure on $\mathbb{R}^n$.
$B_\rho(u; V) = \{v \in V\ ;\ \lVert u - v \rVert < \rho\}$	open ball of radius ρ around $u \in V$; in particular, if $V = \mathbb{R}^n$, then $B_\rho(x_0) = B_\rho(x_0; \mathbb{R}^n)$, $B_\rho = B_\rho(0)$
Re	real part
Im	imaginary part
c, C	generic constants
Cross-references	$(N.x.y)$ refers to formula (x, y) in Chapter N $(x.y)$ within Chapter N refers to formula $(N.x.y)$.