

Classics in Mathematics

Daniel W. Stroock

S. R. Srinivasa Varadhan

Multidimensional Diffusion Processes

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To our parents:
Katherine W. Stroock
Alan M. Stroock
S.R. Janaki
S.V. Ranga Ayyangar

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Frequently Used Notation

I. Topological Notation. Let (X, ρ) be a separable metric space.

- 1) B° is the interior of $B \subseteq X$.
- 2) $\bar{B}$ is the closure of $B \subseteq X$.
- 3) ∂B is the boundary of $B \subseteq X$.
- 4) $\mathcal{B}_X$ is the Borel field of subsets of X .
- 5) $C_b(X)$ is the set of bounded continuous functions $f: X \rightarrow R$.
- 6) $B(X)$ is the set of bounded $\mathcal{B}_X$ -measurable $f: X \rightarrow R$.
- 7) $U_\rho(X)$ is the set of bounded ρ -uniformly continuous $f: X \rightarrow R$.
- 8) $M(X)$ is the set of probability measures on $(X, \mathcal{B}_X)$.
- 9) $\|f\| = \sup_{x \in X} |f(x)|$ for $f \in B(X)$.

II. Special Notation for Euclidean Spaces

- 1) R^d is d -dimensional Euclidean space.
- 2) $|x| = (\sum_{j=1}^d x_j^2)^{1/2}$ for $x \in R^d$.
- 3) $B(x, r) = \{y \in R^d: |x - y| < r\}$.
- 4) $\langle x, y \rangle = \sum_{j=1}^d x_j y_j$ for $x, y \in R^d$.
- 5) $S^{d-1} = \{x \in R^d: |x| = 1\}$.
- 6) $\hat{C}(R^d) = \{f \in C_b(R^d): \lim_{|x| \rightarrow \infty} f(x) = 0\}$.
- 7) $C_0(\mathcal{G})$ is the set of $f \in C_b(\mathcal{G})$ having compact support.
- 8) $C_b^m(\mathcal{G})$ is the set of $f: \mathcal{G} \rightarrow R$ possessing bounded continuous derivatives of order up to and including m .
- 9) $C_b^\infty(\mathcal{G}) = \bigcap_{m=0}^{\infty} C_b^m(\mathcal{G})$.
- 10) $C^\infty(\mathcal{G})$ is the set of $f: \mathcal{G} \rightarrow R$ possessing continuous derivatives of all orders.
- 11) $C_0^\infty(\mathcal{G}) = C^\infty(\mathcal{G}) \cap C_0(\mathcal{G})$.
- 12) $C_b^{m,n}(\mathcal{G})$ for $\mathcal{G} \subseteq [0, \infty) \times R^d$ is the set of $f: \mathcal{G} \rightarrow R$ such that f has m bounded continuous time derivatives and bounded continuous spacial derivatives of order less than or equal to n .
- 13) $L^p(\mathcal{G})$, $1 \leq p \leq \infty$, is the usual L^p -space defined in terms of Lebesgue measure on $\mathcal{G}$.

- 14) $L^p_{\text{loc}}(\mathcal{G})$ is the set of $f: \mathcal{G} \rightarrow R$ (or $\mathbb{C}$) such that $f \in L^p(K)$ for all compact K in $\mathcal{G}$.

III. Path Spaces Notation

- 1) $C(I, R^d)$ for $I \subseteq [0, \infty)$ is the set of R^d -valued functions on I into R^d .
- 2) $\Omega_d(\Omega)$ (see p. 30).
- 3) $\mathcal{M}_t(\mathcal{M})$ (see p. 30).
- 4) $x(t, \omega)$ (see p. 30).

IV. Miscellaneous Notation

- 1) $a \wedge b$ is the smaller of the numbers $a, b \in R$.
- 2) $a \vee b$ is the larger of the numbers $a, b \in R$.
- 3) S_d is the set of symmetric non-negative definite $d \times d$ real matrices.
- 4) S_d^+ is the set of nondegenerate elements of S_d .
- 5) $\|A\|$, where A is a square matrix, and is the operator norm of A .
- 6) $\sigma(\mathcal{C})$, where $\mathcal{C}$ is a collection of subsets of X , and is the smallest σ -algebra over X containing $\mathcal{C}$.
- 7) $\sigma(\mathcal{F})$, where $\mathcal{F}$ is a set of functions on X into a measurable space, and is the smallest σ -algebra over X with respect to which every element of $\mathcal{F}$ is measurable.
- 8) $[\lambda]$, $\lambda \in R$, is the integral part of λ .
- 9) $\xi \sim I_d^s(a, b)$ (see p. 92).