



Progress in Mathematics

Volume 41

Series Editors

Hyman Bass

Joseph Oesterlé

Alan Weinstein

Richard P. Stanley

Combinatorics
and
Commutative Algebra
Second Edition

Birkhäuser
Boston • Basel • Berlin

Richard P. Stanley
Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA 02139

Library of Congress Cataloging-in-Publication Data

Stanley, Richard P., 1944-

Combinatorics and commutative algebra / Richard P. Stanley. -- 2nd ed.

p. cm. -- (Progress in mathematics ; v. 41)

Includes bibliographical references.

ISBN 0-8176-3836-9 (alk. paper). -- ISBN 3-7643-3836-9 (alk. paper)

1. Commutative algebra. 2. Combinatorial analysis. I. Title.

II. Series: Progress in mathematics (Boston, Mass.) ; vol. 41

QA251.3.S72 1996

95-25196

512'.24--dc20

CIP

Printed on acid-free paper

© 1996 Birkhäuser Boston, 2nd ed.

1983 Birkhäuser Boston, 1st ed.

Birkhäuser 

Copyright is not claimed for works of U.S. Government employees.

All rights reserved. No part of this publication may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopying, recording, or otherwise, without prior permission of the copyright owner.

Permission to photocopy for internal or personal use of specific clients is granted by Birkhäuser Boston for libraries and other users registered with the Copyright Clearance Center (CCC), provided that the base fee of \$6.00 per copy, plus \$0.20 per page is paid directly to CCC, 222 Rosewood Drive, Danvers, MA 01923, U.S.A. Special requests should be addressed directly to Birkhäuser Boston, 675 Massachusetts Avenue, Cambridge, MA 02139, U.S.A.

ISBN 0-8176-3836-9

ISBN 3-7643-3836-9

Layout, typesetting by TeXniques, Boston, MA

Printed and bound by Quinn-Woodbine, Woodbine, NJ

Printed in the United States of America

9 8 7 6 5 4 3 2 1

Contents

Preface to the Second Edition	vii
Preface to the First Edition	viii
Notation	ix
Chapter 0: Background	
1. Combinatorics	1
2. Commutative algebra and homological algebra	6
3. Topology	19
Chapter I: Nonnegative Integral Solutions to Linear Equations	
1. Integer stochastic matrices (magic squares)	25
2. Graded algebras and modules	26
3. Elementary aspects of \mathbb{N} -solutions to linear equations	28
4. Integer stochastic matrices again	32
5. Dimension, depth, and Cohen–Macaulay modules	33
6. Local cohomology	36
7. Local cohomology of the modules $M_{\Phi, \alpha}$	38
8. Reciprocity	42
9. Reciprocity for integer stochastic matrices	44
10. Rational points in integral polytopes	45
11. Free resolutions	46
12. Duality and canonical modules	48
13. A final look at linear equations	52
Chapter II: The Face Ring of a Simplicial Complex	
1. Elementary properties of the face ring	53
2. f -vectors and h -vectors of complexes and multicomplexes	54
3. Cohen–Macaulay complexes and the Upper Bound Conjecture	58
4. Homological properties of face rings	60
5. Gorenstein face rings	64
6. Gorenstein Hilbert functions	66
7. Canonical modules of face rings	69
8. Buchsbaum complexes	72
Chapter III: Further Aspects of Face Rings	
1. Simplicial polytopes, toric varieties, and the g -theorem	75
2. Shellable simplicial complexes	78
3. Matroid complexes, level complexes, and doubly Cohen–Macaulay complexes	88
4. Balanced complexes, order complexes, and flag complexes	95

5. Splines	106
6. Algebras with straightening law and simplicial posets	110
7. Relative simplicial complexes	116
8. Group actions	119
9. Subcomplexes	126
10. Subdivisions	127
Problems on Simplicial Complexes and their Face Rings	135
Bibliography	145
Index	161

Preface to the Second Edition

Since the appearance of the first edition many further developments have taken place in the area of “combinatorial commutative algebra.” Perhaps the most interesting advances concern the face ring of a simplicial complex, the subject of Chapter 2. Therefore I have added an additional chapter summarizing new work in this area. It provides strong additional evidence of the felicitous symbiosis between the subjects of combinatorics and commutative algebra. I have also added a collection of exercises taken from a course taught at M.I.T. Chapters 0-2 have been corrected and brought up-to-date in only minor ways.

I am grateful to the staff at Birkhäuser for their help in preparing this new edition. Ann Kostant in particular has been an ideal editor, while Sarah Jaffe has done an excellent job of T_EXing the original text of the first edition and merging the list of references there with the many new references. Finally I wish to thank the numerous persons who have contributed valuable suggestions concerning the material in Chapter 3, including Ron Adin, Louis Billera, Anders Björner, Art Duval, David Eisenbud, Takayuki Hibi, Tony Iarrobino, Gil Kalai, and Christian Peskine.

Richard Stanley
Cambridge, Massachusetts
October 20, 1995

Preface to the First Edition

These notes are based on a series of eight lectures given at the University of Stockholm during April and May, 1981. They were intended to give an overview of two topics from “combinatorial commutative algebra,” viz., (1) solutions to linear equations in nonnegative integers (which is equivalent to the theory of invariants of a torus acting linearly on a polynomial ring), and (2) the face ring of a simplicial complex. In order to give a broad perspective many details and specialized topics have been regrettably omitted. In general, proofs have been provided only for those results which were obscure or inaccessible in the literature at the time of the lectures. The original lectures presupposed considerable background in commutative algebra, homological algebra, and algebraic topology. In order to broaden the accessibility of these notes, Chapter 0 has been prepared with the kind assistance of Karen Collins. This chapter provides most of the background information in algebra, combinatorics, and topology needed to read the subsequent chapters.

I wish to express my gratitude to the University of Stockholm, in particular to Jan-Erik Roos, for the kind invitation to visit in conjunction with the year devoted to algebraic geometry and commutative algebra at the Institut Mittag-Leffler. I am also grateful for the many insightful comments and suggestions made by persons attending the lectures, including Anders Björner, Ralf Fröberg, Christer Lech, and Jan-Erik Roos. Special appreciation goes to Anders Björner for the time-consuming and relatively thankless task of writing up these lecture notes. Finally I wish to thank Maura A. McNiff and Ruby Aguirre for their excellent preparation of the manuscript.

Richard Stanley
Cambridge, Massachusetts
May, 1983

Notation

\mathbb{C}	complex numbers
\mathbb{N}	nonnegative integers
\mathbb{P}	positive integers
\mathbb{Q}	rational numbers
\mathbb{R}	real numbers
\mathbb{Z}	integers
\mathbb{R}^+	nonnegative real numbers
$[n]$	for $n \in \mathbb{N}$, the set $\{1, 2, \dots, n\}$
N -matrix	a matrix whose entries belong to the set N
$N[x]$	polynomials in x whose coefficients belong to the set N
$N[[x]]$	formal power series in x whose coefficients belong to the set N
$\#S$	cardinality of the finite set S
$ \cdot $	cardinality or geometric realization, according to context
$T \subseteq S$	T is a subset of S
$T \subset S$	T is a subset of S and $T \neq S$
$\alpha > 0$	for a vector $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$, this means $\alpha_i > 0$ for all i
k^*	nonzero elements of the field k
kE	vector space over k with basis E
\cong	symbol for isomorphism
\approx	symbol for homeomorphism
\oplus, \amalg	direct sum (of vector spaces or modules)
$\text{im } f$	image $f(M)$ of the homomorphism $f : M \rightarrow N$
$\text{ker } f$	kernel of $f : M \rightarrow N$
$\text{vol } \mathcal{P}$	volume (= Lebesgue measure) of the set $\mathcal{P} \subseteq \mathbb{R}^n$
$\delta_{i,j}$	the Kronecker delta (= 1 if $i = j$, and = 0 if $i \neq j$)