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Richard P. Stanley

Combinatorics and Commutative Algebra

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Preface to the Second Edition

Since the appearance of the first edition many further developments have taken place in the area of "combinatorial commutative algebra." Perhaps the most interesting advances concern the face ring of a simplicial complex, the subject of Chapter 2. Therefore I have added an additional chapter summarizing new work in this area. It provides strong additional evidence of the felicitous symbiosis between the subjects of combinatorics and commutative algebra. I have also added a collection of exercises taken from a course taught at M.I.T. Chapters 0-2 have been corrected and brought up-to-date in only minor ways.

I am grateful to the staff at Birkhäuser for their help in preparing this new edition. Ann Kostant in particular has been an ideal editor, while Sarah Jaffe has done an excellent job of TEXing the original text of the first edition and merging the list of references there with the many new references. Finally I wish to thank the numerous persons who have contributed valuable suggestions concerning the material in Chapter 3, including Ron Adin, Louis Billera, Anders Björner, Art Duval, David Eisenbud, Takayuki Hibi, Tony Iarrobino, Gil Kalai, and Christian Peskine.

Richard Stanley
Cambridge, Massachusetts
October 20, 1995

Preface to the First Edition

These notes are based on a series of eight lectures given at the University of Stockholm during April and May, 1981. They were intended to give an overview of two topics from "combinatorial commutative algebra," viz., (1) solutions to linear equations in nonnegative integers (which is equivalent to the theory of invariants of a torus acting linearly on a polynomial ring), and (2) the face ring of a simplicial complex. In order to give a broad perspective many details and specialized topics have been regretfully omitted. In general, proofs have been provided only for those results which were obscure or inaccessible in the literature at the time of the lectures. The original lectures presupposed considerable background in commutative algebra, homological algebra, and algebraic topology. In order to broaden the accessibility of these notes, Chapter 0 has been prepared with the kind assistance of Karen Collins. This chapter provides most of the background information in algebra, combinatorics, and topology needed to read the subsequent chapters.

I wish to express my gratitude to the University of Stockholm, in particular to Jan-Erik Roos, for the kind invitation to visit in conjunction with the year devoted to algebraic geometry and commutative algebra at the Institut Mittag-Leffler. I am also grateful for the many insightful comments and suggestions made by persons attending the lectures, including Anders Björner, Ralf Fröberg, Christer Lech, and Jan-Erik Roos. Special appreciation goes to Anders Björner for the time-consuming and relatively thankless task of writing up these lecture notes. Finally I wish to thank Maura A. McNiff and Ruby Aguirre for their excellent preparation of the manuscript.

Richard Stanley
Cambridge, Massachusetts
May, 1983

Notation

$\mathbb{C}$	complex numbers
$\mathbb{N}$	nonnegative integers
$\mathbb{P}$	positive integers
$\mathbb{Q}$	rational numbers
$\mathbb{R}$	real numbers
$\mathbb{Z}$	integers
$\mathbb{R}^+$	nonnegative real numbers
$[n]$	for $n \in \mathbb{N}$, the set $\{1, 2, \ldots, n\}$
N-matrix	a matrix whose entries belong to the set N
$N[x]$	polynomials in x whose coefficients belong to the set N
$N[[x]]$	formal power series in x whose coefficients belong to the set N
$\#S$	cardinality of the finite set S
$\lvert \cdot \rvert$	cardinality or geometric realization, according to context
$T \subseteq S$	T is a subset of S
$T \subset S$	T is a subset of S and $T \neq S$
$\alpha > 0$	for a vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$, this means $\alpha_i > 0$ for all i
k^*	nonzero elements of the field k
kE	vector space over k with basis E
$\cong$	symbol for isomorphism
$\approx$	symbol for homeomorphism
$\oplus, \coprod$	direct sum (of vector spaces or modules)
$\operatorname{im} f$	image $f(M)$ of the homomorphism $f : M \rightarrow N$
$\ker f$	kernel of $f : M \rightarrow N$
$\operatorname{vol} \mathcal{P}$	volume (= Lebesgue measure) of the set $\mathcal{P} \subseteq \mathbb{R}^n$
δ_{ij}	the Kronecker delta (= 1 if $i = j$, and = 0 if $i \neq j$)