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# **Special Functions of Mathematical Physics**

A Unified Introduction with Applications

Translated from the Russian by Ralph P. Boas

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