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Solving Ordinary Differential Equations II

Stiff and Differential-Algebraic
Problems

With 129 Figures



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