Undergraduate Texts in Mathematics

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With 117 Figures

Springer

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Preface

This book is about geometry. In particular, it is about the idea of curvature and how it affects the assumptions about and principles of geometry. That being said, I should mention that the word "curvature" does not even appear until the end of the fifth chapter of the book. Before then, it is hidden within the idea of the sum of the angles in a triangle.

In the course of the text, we consider the effects of different assumptions about the sum of the angles in a triangle. The main conceptual tool is the tiling, or tessellation, of the plane. Changing our assumptions on triangles leads to vastly different consequences, which can be seen (literally) in the geometric patterns that arise in tilings.

The result of this point of view is a text that goes in atypical directions for a geometry book. In the process of looking at geometric objects, I bring in the algebra of complex (and hypercomplex) numbers, some graph theory, and some topology. Nevertheless, my intent is to keep the book at an elementary level. The readers of this book are assumed to have had a course in Euclidean geometry (including some analytic geometry) and some algebra, all at the high-school level. No calculus or trigonometry is assumed, except that I occasionally refer to sines and cosines. On the other hand, the book touches on topics that even math majors at college may not have seen. This occurs in Chapter 5, so it is possible to skip some or all of this. But I think that would be a mistake. While the ideas in that chapter are advanced, the mathematical techniques are not. For me, that chapter was the main reason for writing this book.

Here is a brief summary of the contents:

Chapter 1 is an introduction to non-Euclidean geometry. Euclid's axiomatic system is based on five postulates, of which four are reasonably intuitive. The fifth postulate, however, is quite another story. In the process of attempting to prove that this postulate follows from the others and is therefore unnecessary, mathematicians discovered many equivalent formulations. The one that is central to this text is the statement due to Gerolamo Saccheri (1733): *The sum of the angles in a triangle is equal to two right angles.* Non-Euclidean geometry begins with the negation of this statement. Throughout the text we will be exploring the consequences of assuming that the sum of the angles in a triangle is always equal to, always less than, or always greater than two right angles. The last section presents a "proof" due to Saccheri that the sum of the angles in a triangle cannot be greater than two right angles, and a "proof" due to Adrien-Marie Legendre that the sum of the angles cannot be less than two right angles.

Chapter 2 proceeds from the assumption that the angle sum is always 180°. We consider the process of tiling the plane with regular polygons. Section 2.1 sets up the machinery of isometries and transformation groups. In Section 2.2 we find all regular and semiregular tilings. A curious unsuccessful attempt to tile the plane with pentagons leads to the construction of self-similar patterns and leads to a digression on fractals. The last section introduces complex numbers as a tool for studying plane geometry.

In Chapter 3 we start instead with the assumption that the angle sum is always less than 180°. This is the underlying postulate for hyperbolic geometry; it can be illustrated by the Poincaré disc. Using this model leads to the hyperbolic tilings, including those used by M.C. Escher in some of his artwork. The simplest description of isometries in this model uses fractional linear (Möbius) transformations. So in Section 3.3 we apply the arithmetic of complex numbers to the geometry of these transformations.

Chapter 4 uses the assumption of angle sums greater than 180°, which is the postulate underlying elliptic geometries. Section 4.1 is a brief look at the complications this assumption causes. We explore the possibility of more than one line connecting two points, relating this to the geometry of the sphere. In the second section the problem of tiling the sphere leads to an introduction to graphs. We derive Euler's formula. The third section consists of the classification of regular and semiregular tilings of the sphere, and the construction of regular and semiregular convex polyhedra. The last section looks at the geometry of the projective plane and includes a description of the Möbius band as the set of lines in the plane.

The fifth chapter, like the Fifth Postulate, is quite a bit more complicated than the first four. It includes topics not found in most (any?) elementary geometry books. Section 1 contains Cauchy's theorem, which states that closed convex polyhedral surfaces are rigid. Although this is an advanced theorem, the proof is elementary and relies only on properties of polygons and Euler's formula.

Section 5.2 generalizes the construction of complex numbers from Chapter 2 to hypercomplex numbers (quaternions). Using the arithmetic of such numbers, we look at the problem of figuring out the effect of two consecutive rotations of the sphere about different axes. Along the way, some of the basic ideas of algebra show up. The third section describes the notion of curvature for polyhedra and includes a proof of the polyhedral Gauss–Bonnet theorem. Again this advanced theorem turns out to rely only on Euler's formula.

Chapter 6 is a brief, nontechnical, discussion of how all of the ideas of the previous chapters can be blended together into a more general notion of geometry. The sum of the angles is used to quantify the curvature of a piece of (two-dimensional) space. A general curved space, either polyhedral or "smooth," is allowed to have curvature that varies from place to place. Straight lines give way to geodesics. We briefly examine the mysterious behavior of shortest paths on polyhedra. A few final words about space–time and general relativity close the chapter.

The contents of this book can be covered in a one-semester course; on the other hand, it would be easy to spend a lot more time on some of the topics than such a schedule would permit. Some sections are very easy to omit: in particular, the discussion of Möbius transformations (Section 3.3) and the discussion of quaternions (Section 5.2) can be dropped to reduce the difficulty level. Section 2.2 on complex numbers is used in those two sections but not elsewhere.

Finally, a note about proofs and mathematical rigor. I have attempted to be precise, not vague, about technical issues, but I have generally avoided the "theorem–proof" style of exposition. Geometry is a fascinating subject, which many people find exciting and beautiful. It is better not to sterilize it by obscuring the main ideas in Euclidean formalism. On the other hand, some rigor is absolutely essential to the subject. Failure to be careful about geometric arguments has led to a lot of nonsense. In Chapter 1, I present the "proof" of the parallel postulate. It has been my practice, in teaching the course for which this book forms the basis, to begin by presenting this proof, preceded by the warning that it is not correct. I believe that the best way to understand the need for proof in mathematics is to see a really good false proof. (This one is a beauty, due to no less a mathematician than Legendre!) After that, I expect my students to be able to convince each other and me of the truth of claims they make. I also expect them to challenge me if they are not convinced about claims I make. This is the ideal environment for mathematical rigor.

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