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Best Approximation  
in Normed Linear Spaces  
by Elements of Linear Subspaces



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# CONTENTS

	Page
CONTENTS .....	5
PREFACE .....	9
PREFACE TO THE ENGLISH EDITION .....	11
INTRODUCTION .....	13

## Chapter I

BEST APPROXIMATION IN NORMED LINEAR SPACES BY ELEMENTS OF ARBITRARY LINEAR SUBSPACES .....	17
<b>§1. Characterizations of elements of best approximation</b> . . . . .	<b>17</b>
1.1. The first theorem of characterization of elements of best approximation in general normed linear spaces . . . . .	18
1.2. Geometrical interpretation . . . . .	24
1.3. Applications in the spaces $C(Q)$ . . . . .	29
1.4. Applications in the spaces $C_R(Q)$ . . . . .	33
1.5. Applications in the spaces $L^1(T, \nu)$ . . . . .	45
1.6. Applications in the spaces $C^1(Q, \nu)$ and $C_R^1(Q, \nu)$ . . . . .	55
1.7. Applications in the spaces $L^p(T, \nu)$ ( $1 < p < \infty$ ) and in inner product spaces . . . . .	56
1.8. The second theorem of characterization of elements of best approximation in general normed linear spaces . . . . .	58
1.9. Geometrical interpretation . . . . .	67
1.10. Applications and geometrical interpretation in the spaces $C(Q)$ . . . . .	69
1.11. Applications in linear subspaces of the spaces $C(Q)$ . . . . .	75
1.12. Applications in the spaces $L^1(T, \nu)$ . . . . .	83
1.13. Other characterizations of elements of best approximation in general normed linear spaces . . . . .	87
1.14. Orthogonality in general normed linear spaces . . . . .	91
<b>§2. Existence of elements of best approximation</b> . . . . .	<b>93</b>
<b>§3. Uniqueness of elements of best approximation</b> . . . . .	<b>103</b>
3.1. Uniqueness of elements of best approximation in general normed linear spaces . . . . .	103
3.2. Applications in the spaces $C(Q)$ and $C_R(Q)$ . . . . .	117

3.3. Applications in the spaces $L^1(T, \nu)$ and $L_R^1(T, \nu)$ . . . . .	120
3.4. Applications in the spaces $C^1(Q, \nu)$ and $C_R^1(Q, \nu)$ . . . . .	123
<b>§4. <math>k</math>-dimensional <math>\mathfrak{A}_G(x)</math> sets</b> . . . . .	125
4.1. Preliminaries . . . . .	125
4.2. $k$ -semi-Čebyšev linear subspaces in general normed linear spaces . . . . .	126
4.3. Applications in the spaces $C(Q)$ and $C_R(Q)$ . . . . .	131
4.4. Applications in the spaces $L^1(T, \nu)$ , $L_R^1(T, \nu)$ , $C^1(Q, \nu)$ and $C_R^1(Q, \nu)$ . . . . .	133
<b>§5. Interpolative best approximation, best approximation by elements of linear manifolds and their equivalence to best approximation by elements of linear subspaces</b> . . . . .	135
<b>§6. The operators <math>\pi_G</math> and the functionals <math>e_G</math>. Deviations. Elements of <math>\varepsilon</math>-approximation</b> . . . . .	139
6.1. The operators $\pi_G$ . . . . .	140
6.2. The functionals $e_G$ . . . . .	147
6.3. The functionals $e_{G_n}$ for increasing or decreasing sequences $\{G_n\}$ of closed linear subspaces . . . . .	151
6.4. The deviation of a set from a linear subspace . . . . .	156
6.5. Elements of $\varepsilon$ -approximation . . . . .	162

## Chapter II

<b>BEST APPROXIMATION IN NORMED LINEAR SPACES BY ELEMENTS OF LINEAR SUBSPACES OF FINITE DIMENSION</b> .....	165
<b>§1. Characterizations of polynomials of best approximation</b> . . . . .	166
1.1. Preliminary lemmas . . . . .	166
1.2. Characterizations of polynomials of best approximation in general normed linear spaces . . . . .	170
1.3. Applications in the spaces $C(Q)$ , $C_R(Q)$ and $C_0(T)$ . . . . .	178
1.4. The conjugate space of the space $C_E(Q)$ and the extremal points of its unit cell . . . . .	191
1.5. Applications in the spaces $C_E(Q)$ . . . . .	201
1.6. Applications in the spaces $L^1(T, \nu)$ , $L_R^1(T, \nu)$ , $C^1(Q, \nu)$ and $C_R^1(Q, \nu)$ . . . . .	203
<b>§2. Uniqueness of polynomials of best approximation</b> . . . . .	206
2.1. Preliminary lemmas . . . . .	206
2.2. Finite dimensional Čebyšev subspaces in general normed linear spaces . . . . .	210
2.3. Applications in the spaces $C(Q)$ , $C_R(Q)$ , $C_0(T)$ and $L^\infty(T, \nu)$ . . . . .	215
2.4. Applications in the spaces $C_E(Q)$ . . . . .	225
2.5. Applications in the spaces $L^1(T, \nu)$ , $L_R^1(T, \nu)$ , $C^1(Q, \nu)$ and $C_R^1(Q, \nu)$ . . . . .	226
<b>§3. Finite dimensional <math>k</math>-Čebyšev subspaces</b> . . . . .	237
3.1. Finite dimensional $k$ -Čebyšev subspaces in general normed linear spaces . . . . .	238
3.2. Applications in the spaces $C(Q)$ and $C_R(Q)$ . . . . .	240

§4. **Polynomial interpolative best approximation. Best approximation by elements of finite dimensional linear manifolds** . . . . . 242

    4.1. The case of general normed linear spaces . . . . . 242

    4.2. Applications in the spaces  $C(Q)$  and  $C_R(Q)$  . . . . . 244

§5. **The operators  $\pi_G$  and the functionals  $e_G$  for linear subspaces  $G$  of finite dimension** . . . . . 246

    5.1. The operators  $\pi_G$  for linear subspaces  $G$  of finite dimension . . . . . 246

    5.2. The operators  $\pi_{G_n}$  for increasing sequences  $\{G_n\}$  of linear subspaces of finite dimension . . . . . 252

    5.3. The functionals  $e_{G_n}$  for increasing sequences  $\{G_n\}$  of linear subspaces of finite dimension . . . . . 262

§6.  **$n$ -dimensional diameters. Best  $n$ -dimensional secants** . . . . . 268

    6.1. Preliminary lemmas . . . . . 269

    6.2.  $n$ -dimensional diameters . . . . . 274

    6.3. Best  $n$ -dimensional secants . . . . . 282

    6.4. Best  $n$ -dimensional  $\forall$ -secants. Čebyšev centers. Closest points to a set. Best  $n$ -nets. Best  $n$ -coverings . . . . . 287

**Chapter III**

**BEST APPROXIMATION IN NORMED LINEAR SPACES BY ELEMENTS OF CLOSED LINEAR SUBSPACES OF FINITE CODIMENSION** . . . . . 291

§1. **Best approximation by elements of factor-reflexive closed linear subspaces** . . . . . 292

§2. **Best approximation by elements of closed linear subspaces of finite codimension** . . . . . 295

    2.1. Best approximation by elements of closed linear subspaces of finite codimension in general normed linear spaces . . . . . 295

    2.2. Applications in the spaces  $C_R(Q)$  . . . . . 302

    2.3. Applications in the spaces  $L_R^1(T, \nu)$  . . . . . 325

§3. **Best approximation in conjugate spaces by elements of weakly\* closed linear subspaces of finite codimension** . . . . . 333

    3.1. Weakly\* closed Čebyšev subspaces of finite codimension in general conjugate spaces . . . . . 333

    3.2. Applications in the spaces  $L_R^1(T, \nu)^*$  . . . . . 335

    3.3. Applications in the spaces  $C_R(Q)^*$ ,  $L_R^\infty(T, \nu)^*$  and  $((c_0)_R)^*$  . . . . . 339

§4. **The operators  $\pi_G$  and the functionals  $e_G$  for closed linear subspaces  $G$  of finite codimension. Diameters of order  $n$**  . . . . . 350

    4.1. The operators  $\pi_G$  for closed linear subspaces  $G$  of finite codimension . . . . . 350

    4.2. The operators  $\pi_{G^n}$  for decreasing sequences  $\{G^n\}$  of closed linear subspaces of finite codimension . . . . . 353

    4.3. The functionals  $e_{G^n}$  for decreasing sequences  $\{G^n\}$  of closed linear subspaces of finite codimension . . . . . 355

    4.4. Diameters of order  $n$  . . . . . 357

### Appendix I

BEST APPROXIMATION IN NORMED LINEAR SPACES BY ELEMENTS OF NON-LINEAR SETS.....	359
§1. Best approximation by elements of convex sets . . . . .	360
§2. The problem of convexity of Čebyšev sets . . . . .	364
§3. Best approximation by elements of finite dimensional surfaces . . . . .	371
§4. Best approximation by elements of arbitrary sets . . . . .	374

### Appendix II

BEST APPROXIMATION IN METRIC SPACES BY ELEMENTS OF ARBITRARY SETS .....	377
§1. Properties of the sets $\mathfrak{E}_G(x)$ . A characterization of elements of best approximation . . . . .	379
§2. Proximinal sets . . . . .	382
§3. Properties of the mappings $\mathfrak{E}_G$ . . . . .	386
§4. Properties of the mappings $\pi_G$ and of the functionals $e_G$ . . . . .	390
BIBLIOGRAPHY .....	393

## P R E F A C E

*The existing monographs on approximation theory and on the constructive theory of functions (e.g. N. I. Ahiezer [1], J. R. Rice [192], A. F. Timan [250], I. P. Natanson [158] etc.), treat the problems of best approximation in classical style, with the methods and language of the theory of functions, the use of functional analysis being reduced to a few elementary results on best approximation in normed linear spaces and in Hilbert spaces. In contrast to these, the present monograph attempts to give a modern theory of best approximation, using in a consequent manner the methods of functional analysis.*

*From the vast field of best approximation, this monograph presents in more detail the results on best approximation in normed linear spaces by elements of linear subspaces, which today constitute a unified theory. The more general problems of best approximation are exposed, briefly, in Appendices I and II. A glance at the table of contents shows that together with results in general normed linear spaces there are given many applications of them in various concrete spaces.*

*In order to limit the size of the present monograph, we deliberately have omitted some problems related to those treated herein (e.g. applications to extremal problems of the theory of analytic functions, methods of computation of elements of best approximation, the problem of moments, connections with linear programming, etc.). Also, some results related to those presented here are mentioned without proof.*

*Being the first of this kind in the literature, the present monograph is based exclusively on papers published in mathematical*



*journals; the references to these are given in the text after each result. The bibliography given at the end does not aim at being complete, it includes only papers which are effectively quoted in the text.*

*The present monograph is intended for a large circle of mathematicians. Firstly, it is addressed to specialists in approximation theory and the constructive theory of functions, offering to them the methods of functional analysis for the study of these classical domains of mathematics; the necessity and advantages of these methods are shown in the "Introduction". Secondly, it is addressed to those working in functional analysis, offering them an important field of applications. Also, taking into account that in the problems investigated there are combined the methods of functional analysis, geometry, general topology, measure theory and other mathematical disciplines, we hope that the present monograph will be useful to other categories of readers as well (e.g. to specialists in the geometry of convex bodies, etc.).*

*The reader is assumed to know the elements of functional analysis and the mathematical disciplines mentioned above, e.g. within the limits of university courses. However, in order to facilitate the reading and to make the book accessible to University students as well, we have indicated, in connection with the results used (of functional analysis, theory of measure, etc.), a reference to a treatise containing the proof of the respective result; when we have used results which are not to be found in monographs but only in papers published in journals, we have mentioned them in the form of lemmas, giving also their proof.*

*In conclusion, I wish to express my thanks to Miron Nicolescu, member of the Academy, for the invitation to write this monograph and for the constant interest shown during its elaboration. Also, I extend my thanks to my colleagues N. Dinculeanu and C. Foiaş for valuable discussions on the proofs of certain theorems and to V. Klee of the University of Washington in Seattle for some bibliographical indications.*

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