

Saunders Mac Lane  
Ieke Moerdijk

# Sheaves in Geometry and Logic

A First Introduction to Topos Theory



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