

Contents

Preface	xv
Why this book?	xv
How this text is organized	xvi
Some suggestions on how to use this text	xxi
About the enclosed CD	xxii
Exercises and solutions	xxiv
Acknowledgements	xxiv
1 Why you need complex numbers	1
Introduction	1
1.1 First analysis of quadratic equations	1
1.2 <i>Mathematica</i> investigation: quadratic equations	3
Exercises	8
2 Complex algebra and geometry	10
Introduction	10
2.1 Informal approach to ‘real’ numbers	10
2.2 Definition of a complex number and notation	12
2.3 Basic algebraic properties of complex numbers	13
2.4 Complex conjugation and modulus	14
2.5 The Wessel–Argand plane	14
2.6 Cartesian and polar forms	15
2.7 DeMoivre’s theorem	21
2.8 Complex roots	25
2.9 The exponential form for complex numbers	29
2.10 The triangle inequalities	32
2.11 <i>Mathematica</i> visualization of complex roots and logs	33
2.12 Multiplication and spacing in <i>Mathematica</i>	35
Exercises	35
3 Cubics, quartics and visualization of complex roots	41
Introduction	41
3.1 <i>Mathematica</i> investigation of cubic equations	42
3.2 <i>Mathematica</i> investigation of quartic equations	45

3.3	The quintic	51
3.4	Root movies and root locus plots	51
	Exercises	53
4	Newton–Raphson iteration and complex fractals	56
	Introduction	56
4.1	Newton–Raphson methods	56
4.2	<i>Mathematica</i> visualization of real Newton–Raphson	57
4.3	Cayley’s problem: complex global basins of attraction	59
4.4	Basins of attraction for a simple cubic	62
4.5	More general cubics	67
4.6	Higher-order simple polynomials	71
4.7	Fractal planets: Riemann sphere constructions	73
	Exercises	76
5	A complex view of the real logistic map	78
	Introduction	78
5.1	Cobwebbing theory	79
5.2	Definition of the quadratic and cubic logistic maps	80
5.3	The logistic map: an analytical approach	81
5.4	What about $n=3,4,\dots?$	89
5.5	Summary of our root-finding investigations	91
5.6	The logistic map: an experimental approach	91
5.7	Experiment one: $0 < \lambda < 1$	92
5.8	Experiment two: $1 < \lambda < 2$	93
5.9	Experiment three: $2 < \lambda < \sqrt{5}$	93
5.10	Experiment four: $2.45044 < \lambda < 2.46083$	95
5.11	Experiment five: $\sqrt{5} < \lambda < \sqrt{5} + \epsilon$	96
5.12	Experiment six: $\sqrt{5} < \lambda$	96
5.13	Bifurcation diagrams	98
5.14	Symmetry-related bifurcation	100
5.15	Remarks	102
	Exercises	103
6	The Mandelbrot set	105
	Introduction	105
6.1	From the logistic map to the Mandelbrot map	105
6.2	Stable fixed points: complex regions	107
6.3	Periodic orbits	110
6.4	Escape-time algorithm for the Mandelbrot set	114
6.5	<i>MathLink</i> versions of the escape-time algorithm	120
6.6	Diving into the Mandelbrot set: fractal movies	126
6.7	Computing and drawing the Mandelbrot set	129
	Exercises	135
	Appendix: C Code listings	136

Contents	ix
----------	----

7 Symmetric chaos in the complex plane	138
Introduction	138
7.1 Creating and iterating complex non-linear maps	139
7.2 A movie of a symmetry-increasing bifurcation	143
7.3 Visitation density plots	145
7.4 High-resolution plots	146
7.5 Some colour functions to try	146
7.6 Hit the turbos with <i>MathLink!</i>	148
7.7 Billion iterations picture gallery	149
Exercises	154
Appendix: C code listings	155
8 Complex functions	159
Introduction	159
8.1 Complex functions: definitions and terminology	159
8.2 Neighbourhoods, open sets and continuity	163
8.3 Elementary vs. series approach to simple functions	165
8.4 Simple inverse functions	169
8.5 Branch points and cuts	171
8.6 The Riemann sphere and infinity	175
8.7 Visualization of complex functions	176
8.8 Three-dimensional views of a complex function	183
8.9 Holey and checkerboard plots	187
8.10 Fractals everywhere?	189
Exercises	192
9 Sequences, series and power series	194
Introduction	194
9.1 Sequences, series and uniform convergence	194
9.2 Theorems about series and tests for convergence	196
9.3 Convergence of power series	202
9.4 Functions defined by power series	205
9.5 Visualization of series and functions	205
Exercises	207
10 Complex differentiation	208
Introduction	208
10.1 Complex differentiability at a point	209
10.2 Real differentiability of complex functions	211
10.3 Complex differentiability of complex functions	212
10.4 Definition via quotient formula	213
10.5 Holomorphic, analytic and regular functions	214
10.6 Simple consequences of the Cauchy–Riemann equations	214
10.7 Standard differentiation rules	215
10.8 Polynomials and power series	217
10.9 A point of notation and spotting non-analytic functions	220

10.10 The Ahlfors–Struble(?) theorem	221
Exercises	233
11 Paths and complex integration	237
Introduction	237
11.1 Paths	237
11.2 Contour integration	240
11.3 The fundamental theorem of calculus	241
11.4 The value and length inequalities	242
11.5 Uniform convergence and integration	243
11.6 Contour integration and its perils in <i>Mathematica!</i>	244
Exercises	245
12 Cauchy’s theorem	248
Introduction	248
12.1 Green’s theorem and the weak Cauchy theorem	248
12.2 The Cauchy–Goursat theorem for a triangle	250
12.3 The Cauchy–Goursat theorem for star-shaped sets	254
12.4 Consequences of Cauchy’s theorem	255
12.5 <i>Mathematica</i> pictures of the triangle subdivision	259
Exercises	261
13 Cauchy’s integral formula and its remarkable consequences	263
Introduction	263
13.1 The Cauchy integral formula	263
13.2 Taylor’s theorem	265
13.3 The Cauchy inequalities	271
13.4 Liouville’s theorem	271
13.5 The fundamental theorem of algebra	272
13.6 Morera’s theorem	274
13.7 The mean-value and maximum modulus theorems	275
Exercises	275
14 Laurent series, zeroes, singularities and residues	278
Introduction	278
14.1 The Laurent series	278
14.2 Definition of the residue	282
14.3 Calculation of the Laurent series	282
14.4 Definitions and properties of zeroes	286
14.5 Singularities	287
14.6 Computing residues	292
14.7 Examples of residue computations	293
Exercises	299

15 Residue calculus: integration, summation and the argument principle	302
Introduction	302
15.1 The residue theorem	302
15.2 Applying the residue theorem	304
15.3 Trigonometric integrals	305
15.4 Semicircular contours	313
15.5 Semicircular contour: easy combinations of trigonometric functions and polynomials	316
15.6 Mousehole contours	318
15.7 Dealing with functions with branch points	320
15.8 Infinitely many poles and series summation	324
15.9 The argument principle and Rouché's theorem	328
Exercises	335
16 Conformal mapping I: simple mappings and Möbius transforms	338
Introduction	338
16.1 Recall of visualization tools	338
16.2 A quick tour of mappings in <i>Mathematica</i>	340
16.3 The conformality property	347
16.4 The area-scaling property	348
16.5 The fundamental family of transformations	348
16.6 Group properties of the Möbius transform	349
16.7 Other properties of the Möbius transform	350
16.8 More about <code>ComplexInequalityPlot</code>	354
Exercises	355
17 Fourier transforms	357
Introduction	357
17.1 Definition of the Fourier transform	358
17.2 An informal look at the delta-function	359
17.3 Inversion, convolution, shifting and differentiation	363
17.4 Jordan's lemma: semicircle theorem II	366
17.5 Examples of transforms	368
17.6 Expanding the setting to a fully complex picture	372
17.7 Applications to differential equations	373
17.8 Specialist applications and other <i>Mathematica</i> functions and packages	376
Appendix 17: older versions of <i>Mathematica</i>	377
Exercises	379
18 Laplace transforms	381
Introduction	381
18.1 Definition of the Laplace transform	381
18.2 Properties of the Laplace transform	383

18.3	The Bromwich integral and inversion	387
18.4	Inversion by contour integration	387
18.5	Convolutions and applications to ODEs and PDEs	390
18.6	Conformal maps and Efros's theorem	395
	Exercises	398
19	Elementary applications to two-dimensional physics	401
	Introduction	401
19.1	The universality of Laplace's equation	401
19.2	The role of holomorphic functions	403
19.3	Integral formulae for the half-plane and disk	406
19.4	Fundamental solutions	408
19.5	The method of images	413
19.6	Further applications to fluid dynamics	415
19.7	The Navier–Stokes equations and viscous flow	425
	Exercises	430
20	Numerical transform techniques	433
	Introduction	433
20.1	The discrete Fourier transform	433
20.2	Applying the discrete Fourier transform in one dimension	435
20.3	Applying the discrete Fourier transform in two dimensions	437
20.4	Numerical methods for Laplace transform inversion	439
20.5	Inversion of an elementary transform	440
20.6	Two applications to 'rocket science'	441
	Exercises	448
21	Conformal mapping II: the Schwarz–Christoffel mapping	451
	Introduction	451
21.1	The Riemann mapping theorem	452
21.2	The Schwarz–Christoffel transformation	452
21.3	Analytical examples with two vertices	454
21.4	Triangular and rectangular boundaries	456
21.5	Higher-order hypergeometric mappings	463
21.6	Circle mappings and regular polygons	465
21.7	Detailed numerical treatments	470
	Exercises	470
22	Tiling the Euclidean and hyperbolic planes	473
	Introduction	473
22.1	Background	473
22.2	Tiling the Euclidean plane with triangles	475
22.3	Tiling the Euclidean plane with other shapes	481
22.4	Triangle tilings of the Poincaré disc	485
22.5	Ghosts and birdies tiling of the Poincaré disc	490
22.6	The projective representation	497

<i>Contents</i>	xiii
22.7 Tiling the Poincaré disc with hyperbolic squares	499
22.8 Heptagon tilings	507
22.9 The upper half-plane representation	510
Exercises	512
23 Physics in three and four dimensions I	513
Introduction	513
23.1 Minkowski space and the celestial sphere	514
23.2 Stereographic projection revisited	515
23.3 Projective coordinates	515
23.4 Möbius and Lorentz transformations	517
23.5 The invisibility of the Lorentz contraction	518
23.6 Outline classification of Lorentz transformations	520
23.7 Warping with <i>Mathematica</i>	524
23.8 From null directions to points: twistors	529
23.9 Minimal surfaces and null curves I: holomorphic parametrizations	531
23.10 Minimal surfaces and null curves II: minimal surfaces and visualization in three dimensions	535
Exercises	538
24 Physics in three and four dimensions II	540
Introduction	540
24.1 Laplace's equation in dimension three	540
24.2 Solutions with an axial symmetry	541
24.3 Translational quasi-symmetry	543
24.4 From three to four dimensions and back again	544
24.5 Translational symmetry: reduction to 2-D	548
24.6 Comments	550
Exercises	551
Bibliography	553
Index	558