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Galois Cohomology

Translated from the French by Patrick Ion



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Foreword

This volume is an English translation of “Cohomologie Galoisienne”. The original edition (Springer LN5, 1964) was based on the notes, written with the help of Michel Raynaud, of a course I gave at the Collège de France in 1962–1963. In the present edition there are numerous additions and one suppression: Verdier’s text on the duality of profinite groups. The most important addition is the photographic reproduction of R. Steinberg’s “Regular elements of semisimple algebraic groups”, Publ. Math. I.H.E.S., 1965. I am very grateful to him, and to I.H.E.S., for having authorized this reproduction.

Other additions include:

- A proof of the Golod-Shafarevich inequality (Chap. I, App. 2).
- The “résumé de cours” of my 1991–1992 lectures at the Collège de France on Galois cohomology of $k(T)$ (Chap. II, App.).
- The “résumé de cours” of my 1990–1991 lectures at the Collège de France on Galois cohomology of semisimple groups, and its relation with abelian cohomology, especially in dimension 3 (Chap. III, App. 2).

The bibliography has been extended, open questions have been updated (as far as possible) and several exercises have been added.

In order to facilitate references, the numbering of propositions, lemmas and theorems has been kept as in the original 1964 text.

Jean-Pierre Serre
Harvard, Fall 1996

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