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# Semilinear Elliptic Equations for Beginners

Existence Results via  
the Variational Approach

 Springer

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