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(continued following index)

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Dynamics of Evolutionary Equations

With 19 Illustrations



Springer

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*With great fondness and gratitude, we dedicate
this volume to our parents.*

*George P. Sell and Alice O. (Roecker) Sell
and
You Qiwen and Wei Yinmei*

PREFACE

The theory and applications of infinite dimensional dynamical systems have attracted the attention of scientists for quite some time. Dynamical issues arise in equations that attempt to model phenomena that change with time. The infinite dimensional aspects occur when forces that describe the motion depend on spatial variables, or on the history of the motion. In the case of spatially dependent problems, the model equations are generally partial differential equations, and problems that depend on the past give rise to differential-delay equations. Because the nonlinearities occurring in these equations need not be small, one needs good dynamical theories to understand the longtime behavior of solutions.

Our basic objective in writing this book is to prepare an entrée for scholars who are beginning their journey into the world of dynamical systems, especially in infinite dimensional spaces. In order to accomplish this, we start with the key concepts of a *semiflow* and a *flow*. As is well known, the basic elements of dynamical systems, such as the theory of attractors and other invariant sets, have their origins here.

In the applications to partial differential equations, for example, the properties of a semiflow serve as a precise statement of the notion of a well-posed problem, which is a central feature in the study of reaction diffusion equations, nonlinear wave equations, and the Navier-Stokes equations. This concept serves as a road map for finding proper solutions in order to drive into the inner city of dynamics of partial differential equations.

Since a time-varying solution of a partial differential equation can be viewed as a trajectory, or curve, in some Banach space W , this suggests that one should rewrite the equation of motion of this solution as an equation in W . The resulting equation is called an evolutionary equation, be it linear or nonlinear. The main approach in this volume is built around the theory of evolutionary equations. (See Chapters 3 and 4.)

Chapter 4 is an especially important feature of this work. Many aspects of

evolutionary equations are collected here, perhaps for the first time, in book form. One should read this chapter on two levels: as a basic introduction and as a reference source. A good approach is to read it more than once, where one goes deeper as the need arises.

The basic applications to the semiflow theory of the Navier-Stokes equations and other partial differential equations are in Chapters 5 and 6. Several aspects of the modern theories of dynamical systems to linear and nonlinear evolutionary equations, such as the perturbation theory of a saddle point, the reduction principle and center manifold, periodic orbits and invariant manifolds, and inertial manifolds appear in Chapters 7 and 8.

Chapter 1 is a brief essay on the Evolution of Evolutionary Equations, with emphasis on the theory of the longtime dynamics of the solutions of these equations. We have chosen to use some poetic license to keep this chapter short. As a result, we do not include an exhaustive list of references to the literature in this chapter. Additional references do appear in the Commentary sections elsewhere in this volume.

There are a number of general readings that are relevant. First there is the pioneering set of lecture notes by Henry (1981) on dynamical issues of nonlinear partial differential equations. Next there is the book by Temam (1988), which contains an encyclopedic treatment of many applications arising in mechanics and physics. The monograph by Hale (1988) contains valuable information on nonlinear dynamics in infinite dimensions, with applications to partial differential equations and differential equations with time delays. For a very good treatment of the dynamics of functional differential equations, see Hale and Verduyn Lunel (1993). The book by Pazy (1983) contains an excellent treatment of the linear theory of semigroups on Banach spaces. Background information on metric space theory, the geometry of Hilbert spaces, and the theory of linear operators can be found in Naylor and Sell (1982). An extensive bibliography on dynamical systems is available on the World Wide Web; see Sell (2000). The references in this bibliography are updated from time to time.

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CONTENTS

Preface	vii
Chapter 1. The Evolution of Evolutionary Equations	1
Chapter 2. Dynamical Systems: Basic Theory	11
2.1 Semiflows and Nonlinear Semigroups	12
2.1.1 Invariant Sets	15
2.1.2 Limit Sets	16
2.1.3 Semiflows on Function Spaces	19
2.1.4 Ordinary Differential Equations on Banach Spaces	20
2.1.5 Flow Homomorphisms	21
2.2 Compact and κ -Contracting Semiflows	22
2.3 Attractors and Global Attractors	27
2.3.1 Asymptotical Compactness	28
2.3.2 Attractors and Their Properties	29
2.3.3 Stability, Dissipativity, and Absorbing Sets	32
2.3.4 Attractors for κ -Contracting Semiflows	34
2.3.5 Existence of Global Attractors	35
2.3.6 Robustness of Attractors	41
2.3.7 Global Attractors: A Summary	44
2.4 Skew Product Dynamics and Nonautonomous Equations	45
2.5 Singular Semiflows	48
2.6 Exercises	50
2.7 Commentary	53
Chapter 3. Linear Semigroups	61
3.1 C_0 -Semigroups and Infinitesimal Generators	62
3.2 An Illustrative Example	66

3.3	Compact and κ -Contracting C_0 -Semigroups	69
3.4	Hille-Yosida and Lumer-Phillips Theorems	69
3.5	Differentiable Semigroups	73
3.6	Analytic Semigroups and Sectorial Operators	76
	3.6.1 Characterizations of Analytic Semigroups	78
	3.6.2 Construction of Analytic Semigroups	84
3.7	Fractional Powers and Interpolation Spaces	92
3.8	Illustrations	103
	3.8.1 Heat Equation	104
	3.8.2 Linear Parabolic Equations	105
	3.8.3 Stokes Equations	111
	3.8.4 Wave Equation	115
	3.8.5 Schrödinger Equation	119
	3.8.6 The $L^1 - L^\infty$ Regularity Property of Analytic Semigroups	120
3.9	Perturbation Theory	126
3.10	Exercises	128
3.11	Commentary	137
Chapter 4. Basic Theory of Evolutionary Equations		141
4.1	PDEs as Evolutionary Equations	143
Part I: Linear Theory		146
4.2	Solution Concepts and the Variation of Constants Formula	146
	4.2.1 C_0 -Theory	148
	4.2.2 Analytic Theory	152
	4.2.3 Compact Theory and Weak Solutions	161
	4.2.4 The C_0 -Theory, Revisited	171
4.3	Linear Skew Product Semiflows	172
4.4	Perturbations of Analytic Semigroups	175
	4.4.1 A Basic Theorem	177
	4.4.2 Topological Issues	185
	4.4.3 Strong Solutions	188
4.5	Exponential Dichotomies: Existence and Robustness	190
	4.5.1 Exponential Dichotomy	192
	4.5.2 Inhomogeneous Equations	203
	4.5.3 Discrete Inhomogeneous Equations	210
	4.5.4 Robustness Theorems	216
Part II: Nonlinear Theory		221
4.6	Well-Posed Problems: C_0 -Theory	224
	4.6.1 Local Existence and Uniqueness Theorems	225

4.6.2	Maximally Defined Solutions	228
4.6.3	Continuous Dependence of Solutions	229
4.6.4	Construction of the Nonlinear Semiflow	231
4.7	Well-Posed Problems: Analytic Theory	232
4.7.1	Mild Solutions	233
4.7.2	Strong Solutions	234
4.7.3	Maximally Defined Solutions	235
4.7.4	Continuous Dependence of Solutions	236
4.7.5	Construction of the Nonlinear Semiflow	237
4.7.6	Extension of the Semiflow	239
4.8	Regularity and Compactness Properties	244
4.8.1	Compactness Properties	244
4.8.2	Regularity in Space and Time	246
4.9	The Linearized Equation	247
4.9.1	Differentiability of Mild Solutions	249
4.9.2	The Linear Skew Product Semiflow, Revisited	251
4.10	Exercises	252
4.11	Commentary	261
Chapter 5. Nonlinear Partial Differential Equations		267
5.1	Reaction Diffusion Equations	269
5.1.1	The Chafee-Infante Equations	269
5.1.2	Systems of Equations and Inhomogeneous Boundary Conditions	281
5.1.3	Partly Dissipative Systems	283
5.2	Nonlinear Wave Equations	284
5.2.1	Abstract Nonlinear Wave Equations	285
5.2.2	Nonlinear Damped Beam Equation	293
5.2.3	Nonlinear Wave Equations with Local Damping	299
5.3	Equations of Convection	313
5.3.1	Construction of the Semiflow	314
5.3.2	Global Attractor in V	318
5.4	Kuramoto-Sivashinsky Equation	319
5.4.1	The Anisymmetric Case	321
5.4.2	The General Case	327
5.5	Cahn-Hilliard Equation	328
5.5.1	Construction of the Semiflow	329
5.5.2	Attractors for the Cahn-Hilliard Equation	335
5.6	Exercises	339
5.7	Commentary	348
Chapter 6. Navier-Stokes Dynamics		359
6.1	Formulation as a Nonlinear Evolutionary Equation	361

6.1.1	The Stokes Operator, Revisited	362
6.1.2	The Nonlinearity	364
6.2	Bubnov-Galerkin Approximations	369
6.3	Weak Solutions	373
6.3.1	The Leary-Hopf Theory	374
6.3.2	Generalized Weak Solutions	389
6.3.3	The Uniqueness Problem	395
6.4	Strong Solutions	396
6.4.1	Two-Dimensional Theory	400
6.4.2	Three-Dimensional Theory	401
6.4.3	The Global Regularity Problem	403
6.4.4	The Linearized Equation	406
6.4.5	Higher Regularity of Solutions	413
6.5	Navier-Stokes Dynamics: Global Attractors	416
6.5.1	Two-Dimensional Theory	416
6.5.2	Three-Dimensional Theory	419
6.5.3	Nonautonomous Problems	424
6.6	The Kwak Transformation	426
6.7	Related Nonlinear Systems	430
6.7.1	Inhomogeneous Boundary Conditions	431
6.7.2	Bénard Convection	433
6.7.3	Chemically Reacting Flows	434
6.8	Proofs for the Bubnov-Galerkin Approximations	436
6.9	Exercises	442
6.10	Commentary	450
Chapter 7. Major Features of Dynamical Systems		457
7.1	Local Dynamics Near an Equilibrium	460
7.1.1	Unstable and Stable Manifold Theorems	464
7.1.2	Center Manifold Theorem	473
7.1.3	Perturbation Theory for Equilibria	481
7.2	Dynamics of Gradient Systems	483
7.3	Behavior Near a Periodic Orbit	488
7.4	Invariant Manifolds	490
7.4.1	Statement of Theorems	496
7.4.2	Local Coordinates Near M	503
7.4.3	The Dynamics on M	505
7.4.4	Perturbed Dynamics Near M	513
7.4.5	Proofs of Theorems	540
7.5	Applications	541
7.5.1	The Couette-Taylor Flow	545
7.5.2	The Bobnov-Galerkin Approximations	548
7.6	Nonautonomous Problems	553
7.7	Other Topics of Dynamical Systems	553

7.7.1	Differential Delay Equations. Functional Differential Equations	553
7.7.2	Bifurcation Theory	553
7.7.3	Ergodic Theory	553
7.7.4	Dimension Theory	554
7.7.5	Minimal Set. The Basic Building Block	556
7.7.6	Singular Perturbations	557
7.7.7	Approximation Dynamics	560
7.7.8	Hamiltonian Systems	562
7.8	Exercises	563
7.9	Commentary	565
Chapter 8. Inertial Manifolds: The Reduction Principle		569
8.1	Introduction	570
8.2	The Lyapunov-Perron Method	575
8.3	Existence of Inertial Manifolds: Spectral Gap Condition	577
8.4	Exponential Attraction of Inertial Manifolds	582
8.5	Smoothness of Inertial Manifolds	585
8.6	Applications of Inertial Manifold Theorems	589
8.7	Open Problems for Inertial Manifolds	591
8.8	Commentary	592
Appendices: Basics of Functional Analysis		593
A	Banach Spaces and Fréchet Spaces	593
B	Function Spaces and Sobolev Imbedding Theorems	604
C	Calculus of Vector-Valued Functions	613
D	Basic Inequalities	621
E	Commentary	626
Bibliography		629
Notation Index		655
Subject Index		659