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James B. Seaborn

Hypergeometric Functions and Their Applications

With 59 Illustrations

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Series Preface

Mathematics is playing an ever more important role in the physical and biological sciences, provoking a blurring of boundaries between scientific disciplines and a resurgence of interest in the modern as well as the classical techniques of applied mathematics. This renewal of interest, both in research and teaching, has led to the establishment of the series: *Texts in Applied Mathematics (TAM)* .

The development of new courses is a natural consequence of a high level of excitement on the research frontier as newer techniques, such as numerical and symbolic computer systems, dynamical systems, and chaos, mix with and reinforce the traditional methods of applied mathematics. Thus, the purpose of this textbook series is to meet the current and future needs of these advances and encourage the teaching of new courses.

TAM will publish textbooks suitable for use in advanced undergraduate and beginning graduate courses, and will complement the *Applied Mathematical Sciences (AMS)* series, which will focus on advanced textbooks and research level monographs.

Preface

A wide range of problems exists in classical and quantum physics, engineering, and applied mathematics in which special functions arise. The procedure followed in most texts on these topics (e.g., quantum mechanics, electrodynamics, modern physics, classical mechanics, etc.) is to formulate the problem as a differential equation that is related to one of several special differential equations (Hermite's, Bessel's, Laguerre's, Legendre's, etc.). The corresponding special functions are then introduced as solutions with some discussion of recursion formulas, orthogonality relations, asymptotic expressions, and other properties as appropriate. In every instance, the reader is referred to a standard text on applied mathematical methods for more detail. This is all very reasonable and proper.

Each special function can be defined in a variety of ways and different authors choose different definitions (Rodrigues formulas, generating functions, contour integrals, etc.). Whatever the starting definition, it is usually shown to be expressible as a series, because this is frequently the most practical way to obtain numerical values for the functions. Also it is often shown—or at least stated—that the special function can be expressed in terms of some generalized hypergeometric function.

In this book, we follow a different track. Each special function arises in one or more physical contexts as a solution of a differential equation that can be transformed into the hypergeometric equation (or its confluent form). The special function is then *defined* in terms of a generalized hypergeometric function. From this definition, many of the interesting and important properties encountered in standard upper level textbooks (recursion formulas, the generating function, orthogonality relations, Rodrigues formula, asymptotic expressions, and various series and integral representations) are derived and the equivalence of this definition to other definitions is established.

This approach is interesting, and it is instructive to see that most of the special functions encountered in applied mathematics have a common root in their relation to the hypergeometric function. The reader may notice that derivations are not always carried out in the simplest or the most straightforward manner. This is usually intentional—a consequence of a deliberate choice made in favor of furnishing the clearest and most direct connections between the functions of applied mathematics and the hypergeometric function rather than of finding the most elegant path to a given result.

In most cases, I have not introduced a mathematical topic until it is necessary for the discussion. For example, complex analysis is not really needed until we begin to look at alternate forms and integral representations of the special functions in Chapter 9. So I have deferred this subject until Chapters 7 and 8 with a few appropriate reminders along the way that it is coming.

Also concerning the mathematics, I have reviewed some of the fundamental notions from calculus (function, continuity, convergence, etc.), which a student may not remember very clearly, but I have tried to hold this kind of review to a minimum. In a few instances, to provide some insight without getting too heavily bogged down in the mathematics, I have used heuristic arguments to establish a desired result.

The range of this book is intentionally rather narrow. There are many interesting and useful topics (conformal mapping, Sturm–Liouville theory, Green's functions, to name a few) which are related to those I have discussed, but which are outside the scope of the task I have in mind.

In writing this book I have assumed that the reader has completed two or three semesters of calculus and has some knowledge of Schrödinger's equation (perhaps, from a course in modern physics or an introductory course in quantum mechanics). Courses at the intermediate level in classical mechanics and electromagnetism are also desirable, but not essential. The book should be accessible to a reader with this minimum preparation. A student who has completed the intermediate courses in an undergraduate physics or engineering curriculum would have a much greater appreciation for the subject matter treated here.

This is all well-plowed ground and I am grateful to those who have worked these fields before me. There are many excellent books on mathematical analysis and methods of mathematical physics, and I have profited greatly from a number of them. Listed in the Bibliography are those that have been most helpful to me in writing this book.

I should like to express my deep gratitude to Professor Gerald Speisman who first excited my interest in this subject a long time ago. Finally, I wish to thank the anonymous reviewers for very helpful comments and suggestions.

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