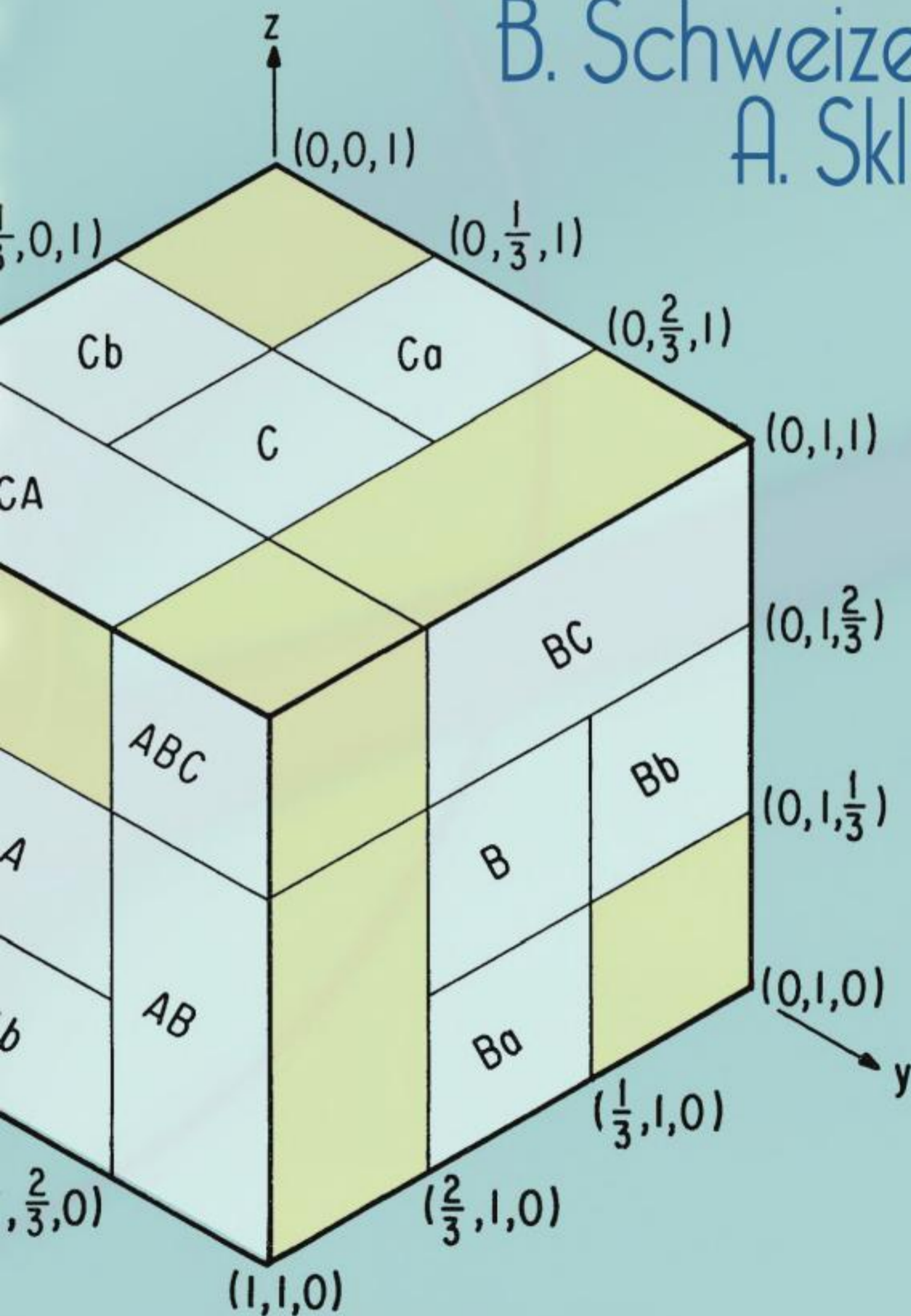


# Probabilistic Metric Spaces

B. Schweizer and  
A. Sklar



*Copyright*

Copyright © 1983 by Elsevier Science Publishing Co., Inc.  
Copyright © 2005 by B. Schweizer and A. Sklar  
All rights reserved.

*Bibliographical Note*

This Dover edition, first published in 2005, is an unabridged republication of the work first published by Elsevier Science Publishing Co., Inc., in 1983. A new Preface to the Dover edition, list of Errata, Notes, and Supplementary References have been added.

*International Standard Book Number: 0-486-44514-3*

Manufactured in the United States by Courier Corporation  
44514302  
[www.doverpublications.com](http://www.doverpublications.com)

# Contents

Preface to the Dover Edition	vii
Preface	ix
Special Symbols	xiii
<b>Chapter 1. Introduction and Historical Survey</b>	<b>1</b>
1.0. Introduction	1
1.1. Beginnings	2
1.2. Menger, 1942	3
1.3. Wald, 1943	4
1.4. Developments, 1956–1960	5
1.5. Some Examples	7
1.6. Šerstnev, 1962	9
1.7. Random Metric Spaces	10
1.8. Topologies	12
1.9. Tools	14
1.10. Postscript	17
<b>Chapter 2. Preliminaries</b>	<b>18</b>
2.1. Sets and Functions	18
2.2. Functions on Intervals	22
2.3. Probabilities, Integrals, Random Variables	24
2.4. Binary Operations	27
<b>Chapter 3. Metric and Topological Structures</b>	<b>30</b>
3.1. Metric and Related Spaces	30
3.2. Isometries, Homotheties, Metric Transforms	33

## Contents

3.3. Betweenness	36
3.4. Minkowski Metrics	37
3.5. Topological Structures	38
<b>Chapter 4. Distribution Functions</b>	<b>43</b>
4.1. Spaces of Distribution Functions	43
4.2. The Modified Lévy Metric	45
4.3. The Space of Distance Distribution Functions	48
4.4. Quasi-Inverses of Nondecreasing Functions	49
<b>Chapter 5. Associativity</b>	<b>54</b>
5.1. Associative Binary Operations	54
5.2. Generators and Ordinal Sums	55
5.3. Associative Functions on Intervals	57
5.4. Representation of Archimedean Functions	62
5.5. Triangular Norms, Additive and Multiplicative Generators	65
5.6. Examples	70
5.7. Conorms and Composition Laws	73
5.8. Open Problems	76
<b>Chapter 6. Copulas</b>	<b>78</b>
6.1. Fundamental Properties of $n$ -Increasing Functions	78
6.2. Joint Distribution Functions, Subcopulas, and Copulas	82
6.3. Copulas and $t$ -Norms	85
6.4. Dual Copulas	88
6.5. Copulas and Random Variables	89
6.6. Margins of Copulas	92
6.7. Open Problems	93
<b>Chapter 7. Triangle Functions</b>	<b>96</b>
7.1. Introduction	96
7.2. The Operations $\tau_{T,L}$	99
7.3. The Operations $\tau_{T^*,L}$	104
7.4. The Operations $\sigma_{C,L}$	106
7.5. The Operations $\rho_{C,L}$	109
7.6. Derivability and Nonderivability from Functions of Random Variables	111
7.7. Duality	114
7.8. The Conjugate Transform	117
7.9. Open Problems	120

<b>Chapter 8. Probabilistic Metric Spaces</b>	<b>124</b>
8.1. Probabilistic Metric Spaces in General	124
8.2. Transformed Triangle Inequalities and Derived Metrics	127
8.3. Equilateral Spaces	131
8.4. Simple Spaces	132
8.5. Ellipse $m$ -Metrics and Hysteresis	133
8.6. $\alpha$ -Simple Spaces	138
8.7. Best-Possible Triangle Inequalities	140
8.8. Open Problems	141
<b>Chapter 9. Random Metric Spaces</b>	<b>142</b>
9.1. $E$ -Spaces	142
9.2. Pseudometrically Generated Spaces, Sherwood's Theorem	145
9.3. Random Metric Spaces	149
9.4. The Probability of the Triangle Inequality	152
9.5. $W$ -Spaces	155
9.6. Open Problems	156
<b>Chapter 10. Distribution-Generated Spaces</b>	<b>157</b>
10.1. Introduction	157
10.2. Consistency, Triangle Inequalities	159
10.3. $C$ -Spaces	161
10.4. Homogeneous and Semihomogeneous $C$ -Spaces	165
10.5. Moments and Metrics	168
10.6. Normal $C$ -Spaces	171
10.7. Moments in Normal $C$ -Spaces	172
10.8. Open Problems	174
<b>Chapter 11. Transformation-Generated Spaces</b>	<b>175</b>
11.1. Transformation-Generated Spaces	175
11.2. Measure-Preserving Transformations	178
11.3. Mixing Transformations	181
11.4. Recurrence	183
11.5. $E$ -Processes: The Case of Markov Chains	186
11.6. Open Problems	189
<b>Chapter 12. The Strong Topology</b>	<b>191</b>
12.1. The Strong Topology and Strong Uniformity	191
12.2. Uniform Continuity of the Distance Function	195
12.3. Examples	198

## Contents

12.4. The Probabilistic Diameter	200
12.5. Completion of Probabilistic Metric Spaces	202
12.6. Contraction Maps	203
12.7. Product Spaces	208
12.8. Countable Products	212
12.9. The Probabilistic Hausdorff Distance	214
12.10. Discernibility Relations	215
12.11. Open Problems	217
<b>Chapter 13. Profile Functions</b>	<b>219</b>
13.1. Profile Closures	219
13.2. Distinguishability	222
<b>Chapter 14. Betweenness</b>	<b>226</b>
14.1. Wald Betweenness	226
14.2. Transform Betweenness	230
14.3. Menger Betweenness	231
14.4. Probabilistic Betweenness	233
14.5. Open Problems	235
<b>Chapter 15. Supplements</b>	<b>236</b>
15.1. Probabilistic Normed Spaces	236
15.2. Probabilistic Inner Product Spaces	240
15.3. Probabilistic Topologies	242
15.4. Probabilistic Information Spaces	246
15.5. Generalized Metric Spaces	249
<b>References</b>	<b>253</b>
<b>Index</b>	<b>267</b>
<b>Errata</b>	<b>277</b>
<b>Notes</b>	<b>281</b>
<b>Supplementary References</b>	<b>305</b>