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Spaces of Vector-Valued Continuous Functions

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INTRODUCTION

Let X be a Hausdorff completely regular space and E be a Hausdorff locally convex topological vector space. Then C(X;E) denotes the linear space of the continuous functions on X with values in E; in the scalar case (i.e. if E is $\mathbb{R}$ or $\mathbb{C}$) we simply write C(X).

The purpose of these notes is to characterize locally convex properties of C(X;E) by means of topological properties of X and of locally convex properties of E.

The scalar case has already been developed in [20] and it is quite convenient to recall briefly its contents. Its chapter I deals with the ultrabornological (resp. bornological; barrelled; quasi-barrelled) space associated to E. One finds in its chapter II a description of the realcompactification υX of X as well as the definition of the space $C_p(X)$, i.e. C(X) endowed with the most general locally convex topology of uniform convergence on subsets of υX. In its chapter III, the ultrabornological (resp. bornological; barrelled; quasi-barrelled) space associated to $C_p(X)$ is characterized. This gives of course a necessary and sufficient condition for $C_p(X)$ to have that property. Its chapter IV concerns conditions of separability or of weak-compactness in $C_p(X)$. Finally its chapter V gives an attempt to study the vector-valued case : it deals with the space $C_s(X;E)$, i.e. C(X;E) endowed with the simple or pointwise topology.

Since the publication of [20], its chapters I to IV have got few complementary results. It is not the case of its chapter V to which significant results have been added. This is due mostly to A. Defant, W. Govaerts, R. Hollstein, J. Mendoza Casas, J. Mujica, myself, ... Now it is possible to say that large parts of the study of the $C_p(X;E)$ spaces are settled.

The aim of these notes is to present these new results as a complement to the chapters I to IV of [20].

The chapter I contains a description of the spaces $C_p(X;E)$, i.e. C(X;E) endowed with the most general locally convex topology of uniform convergence on subsets of υX. The notion of the support of an absolutely convex subset of $C(X;E)$ is studied in the chapter II. The dual of $C_p(X;E)$ is characterized in the chapter III. The chapter IV gives most of the known results dealing with the $C_p(X;E)$ spaces and the ultrabornological, bornological, barrelled, quasi-barrelled and (DF) properties. Finally a special link in between the bounded and the sequentially continuous linear functionals on $C_p(X;E)$ is introduced in the chapter V.

It is a pleasure to thank Mrs. V. Berwart-Verbruggen who has done the typing very carefully. I am grateful to the editors for accepting these notes for publication in the Lecture Notes in Mathematics.

J. SCHMETS

N.B. When a reference is indicated by [20;.], it means the reference [..] of [20].

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