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Hyperfunctions and Harmonic Analysis on Symmetric Spaces

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Henrik Schlichtkrull
Princeton, New Jersey

March, 1984

Til Birgitte

Introduction

The purpose of this book is to give an exposition of the application of hyperfunction theory and microlocal analysis to some important problems in harmonic analysis of symmetric spaces.

The theory of hyperfunctions generalizes that of distributions in the sense that while distributions are linear functionals on C^∞-functions, hyperfunctions can be thought of as linear functionals on the smaller space of analytic functions. For the study of partial differential equations with analytic coefficients this concept is extremely useful. Microlocal analysis is the study (via the tangent space) of the local properties of solutions to systems of such equations.

The book consists of two parts. In the first part (Chapters 1 and 2), which is expository, we give an introduction to hyperfunctions, microlocal analysis, and applications of this theory to the study of systems of partial differential equations with regular singularities. We give very few proofs. As for the main results (Theorems 2.3.1 and 2.3.2), we illustrate the technique of proof via an important example (Section 2.4).

In the second part, we apply the results from the first part to symmetric spaces. Here we give full proofs of all results (with one exception, cf. below); except for certain standard results from the theory of semisimple Lie groups (stated in Chapter 3), this part of the book is self contained (that is, modulo Chapters 1 and 2).

There are two main results that we prove in the second part of the book, concerning respectively a Riemannian symmetric space and a semisimple symmetric space.

Let X be a Riemannian symmetric space of the noncompact type and let $\mathbb{D}(X)$ be the algebra of differential operators on X invariant under all isometries of X. The first result (Corollary 5.4.4) states that every function on X which is an eigenfunction for each operator in $\mathbb{D}(X)$ can be represented by a hyperfunction on the boundary of X via an integral formula similar to the classical Poisson integral for the unit disk. This result, the proof of which comprises Chapters 4 and 5, was conjectured by S. Helgason (1970,[c]) and proved by M. Kashiwara, A. Kowata, K. Minemura, K. Okamoto, T. Oshima and M. Tanaka (1978, Kashiwara et al. [a]) by employing the techniques of microlocal analysis to study the boundary behavior of the eigenfunction to $\mathbb{D}(X)$. This is done by imbedding X into a compact analytic manifold such that the differential operators in $\mathbb{D}(X)$ have regular singularities along the boundary of X (Theorem 4.3.1). The theory from the first part of the book then ensures that the eigenfunctions have certain "boundary values" (or Cauchy data), which are hyperfunctions on the boundary. It is then proved that by taking the Poisson integral of one of these boundary values we recover the original eigenfunction on X (Theorem 5.4.2).

However, in order for the above proof to work, the eigenvalues for the operators in $\mathbb{D}(X)$ have to satisfy a certain regularity assumption (to ensure that no logarithmic terms appear in the process of taking the boundary values). In order to prove Helgason's conjecture for the remaining singular eigenvalues, more refined methods are needed. It is for this most general statement of the conjecture (Theorem 5.4.3) that we make an omission of proof.

In Chapter 6 a generalization of Helgason's conjecture is presented. In the compactification of X (which is known as the maximal Satake-Furstenberg compactification) the so-called boundary of X is in fact only one part of the boundary. The boundary has in general several other "components", and it is natural to represent the eigenfunctions on X also as Poisson integrals of their hyperfunction boundary values on these components (Theorem 6.3.3).

One of the features of the theory of differential equations with regular singularities is that it enables us to derive asymptotic expansions of solutions in the vicinity of the regular singular points. We illustrate this technique by deriving asymptotic expansions of the spherical functions on the Riemannian symmetric space (Theorems 5.3.2 and 6.3.4). These asymptotic expansions (though not in the form of Theorem 6.3.4) were originally derived by Harish-Chandra.

The second main result, concerning a semisimple symmetric space, is proved in Chapters 7 and 8 by using the same technique as was employed in Chapter 6. Let G/H be a semisimple symmetric space (that is, G a semisimple connected Lie group and H a subgroup which is the identity component of the set of fixed points for some involutive autohorphism of G). In the harmonic analysis of G/H one wants to determine the closed subspaces of $L^2(G/H)$ on which G acts irreducibly in the regular representation (that is, the representation of G on $L^2(G/H)$ by left translations) - the so-called discrete series for $L^2(G/H)$. This problem was attacked by M. Flensted-Jensen, who constructed a family of functions on G/H (cf. Section 8.3), which he conjectured to be square integrable (1979, [c]). These functions are eigenfunctions for the invariant differential operators on G/H , and in the "generic" range of the eigenvalue, he proved the square integrability. The conjecture (Theorem 8.3.1) was settled (affirmatively) by T. Oshima (1980, unpublished - cf. Oshima and Matsuki [b]). The proof consists of an application of the theory of regular singularities to derive asymptotic expansions and hence growth estimates for Flensted-Jensen's functions.

The requirements on the part of the reader are as follows. For the hyperfunction theory some familiarity with complex functions of several variables is desirable. However, since this part of the book is expository no deep knowledge is necessary, unless the reader wants to consult the references for proofs. For the applications to symmetric spaces the reader has to be acquainted with some Lie group theory, as for instance is offered in the books Helgason [j] or Wallach [a]. See also Chapter 3 for a more detailed description of the necessary prerequisites.

This book contains several new results. As for the two main results mentioned above, however, the contribution of the author is solely expository. The author's main original contributions are to be found in Chapter 6. Each chapter is concluded with a short section of notes, giving the origin of the theory described in that chapter, with references to the bibliography, which is in the back of the book. The references for the main theorems are Kashiwara and Oshima [a], Oshima and Sekiguchi [a], Oshima [a], Kashiwara et al. [a], Flensted-Jensen [c] and Oshima [c].

Notation.

$\mathbb{R}$ = field of real numbers, $\mathbb{R}_+ = \{t \in \mathbb{R} \mid t \geq 0\}$,

$\mathbb{C}$ = field of complex numbers ,

$\mathbb{Z}$ = ring of integers, $\mathbb{Z}_+ = \mathbb{Z} \cap \mathbb{R}_+$,

$\mathbb{N}$ = set of positive integers.

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