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Classification Theory of Riemann Surfaces

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Preface

The purpose of the present monograph is to systematically develop a classification theory of Riemann surfaces. Some first steps will also be taken toward a classification of Riemannian spaces.

Four phases can be distinguished in the chronological background: the type problem; general classification; compactifications; and extension to higher dimensions.

The type problem evolved in the following somewhat overlapping steps: the Riemann mapping theorem, the classical type problem, and the existence of Green's functions. The Riemann mapping theorem laid the foundation to classification theory: there are only two conformal equivalence classes of (noncompact) simply connected regions. Over half a century of efforts by leading mathematicians went into giving a rigorous proof of the theorem: RIEMANN, WEIERSTRASS, SCHWARZ, NEUMANN, POINCARÉ, HILBERT, WEYL, COURANT, OSGOOD, KOEBE, CARATHÉODORY, MONTEL.

The classical type problem was to determine whether a given simply connected covering surface of the plane is conformally equivalent to the plane or the disk. The problem was in the center of interest in the thirties and early forties, with AHLFORS, KAKUTANI, KOBAYASHI, P. MYRBERG, NEVANLINNA, SPEISER, TEICHMÜLLER and others obtaining incisive specific results. The main problem of finding necessary and sufficient conditions remains, however, unsolved.

At the end of his dissertation RIEMANN had already referred to the significance of the existence of the Green's function. This aspect gave rise to a generalization which chronologically ran somewhat parallel to the classical type problem: finding tests for the class O_G of parabolic surfaces characterized by the nonexistence of Green's functions. The class was studied by P. MYRBERG and explicit criteria were established by AHLFORS, NEVANLINNA, LAASONEN, WITTICH, and LE-VAN.

For plane regions this generalized type problem formed the bridge to the classical theory of SZEGÖ, NEVANLINNA, FROSTMAN, and others on capacities of point sets. In particular O_G turned out to be the class of regions whose boundaries have vanishing logarithmic capacity, Schwarz's harmonic measure, or Fékete's transfinite diameter. Moreover $R \in O_G$ was necessary and sufficient for R to possess Evans-Selberg potentials.

The present monograph will only lightly touch upon these important early developments of classification theory. We start with the second aspect of the evolution, the general classification, which today continues at an ever increasing pace. It was inaugurated in 1948 with the introduction of the class O_{AD} of surfaces without nonconstant AD-functions, i.e. analytic functions with finite Dirichlet integrals. Such surfaces were said to have "hebbar" boundaries, in reference to their behavior as closed surfaces in significant function-theoretic respects (see Introduction and Bibliography). At the Trondheim Congress in 1949 a systematic array of null-classes, together with current notation, was introduced. Penetrating results on function-theoretic null sets related to several such classes were obtained in 1950 by AHLFORS and BEURLING. In 1954 the study of boundary components in classification theory was initiated by the introduction of their capacities.

During the two decades since the beginning of the general classification theory the subject has grown in depth and breadth into one of the major branches of function theory. The main achievements are due to AHLFORS, BEURLING, CONSTANTINESCU, CORNEA, HEINS, KAMETANI, KURAMOCHI, KURODA, KUSUNOKI, LEHTO, MARDEN, MATSUMOTO, A. MORI, S. MORI, L. MYRBERG, P. MYRBERG, NEVANLINNA, NOSHIRO, OHTSUKA, OIKAWA, OZAWA, PARREAU, PFLUGER, RAO, RODIN, ROYDEN, TÔKI, TSUJI, VIRTANEN, YÛJÔBÔ, among others. For a complete list of workers in the field we refer the reader to the Author Index and the Bibliography.

Although capacities of subboundaries are useful especially in the study of plane regions, detailed information on ideal boundaries can only be obtained by compactifying the surface. The mode of compactification depends on the class of functions under consideration. For the class HD of harmonic functions with finite Dirichlet integrals ROYDEN introduced in 1952 the compactification now bearing his name. For the class HB of bounded harmonic functions the Wiener compactification proved to be the most fruitful choice. The Royden and Wiener compactifications can also be described as corresponding to the solution of the Dirichlet problem by Dirichlet's principle or by Perron's method.

The most recent aspects of the theory of compactification started in 1962 with the discovery by KURAMOCHI of surfaces carrying distinguished minimal points on their boundaries. The work was continued in the authoritative treatise of Constantinescu-Cornea in 1963. The current compactification theory as it appears in the present monograph is of the recent vintage of 1966, much of it previously unpublished.

Classification of Riemannian spaces of higher dimensions is the latest facet of the theory. Although only in its infancy it has already produced surprising phenomena.

From the chapter and section headings the reader may obtain an over-all view of the plan of the book. Broadly speaking, regular functions are treated first, then those with logarithmic singularities. Among regular functions the analytic functions precede the harmonic functions; in each category those with finite Dirichlet integrals are discussed first. One denotes by AB and AD the classes of analytic bounded or Dirichlet finite functions, by HB and HD the corresponding classes of harmonic functions, and by O_{AD}, e.g., the class of surfaces without nonconstant AD-functions. The resulting scheme O_{AD}, O_{AB}, O_{HD}, O_{HB}, O_G roughly corresponds to the decreasing "magnitude" of the null classes. Treating O_{AD} at the beginning of the book also has the advantage of first encountering the numerous concrete properties that are associated with O_{AD}, more than with any other class. Finally, starting with O_{AD} somewhat follows the historical development of general classification theory.

A more detailed description of the book is given in brief surveys at the beginning of each chapter and section. For a first orientation of the nonexpert we also give, in the Introduction, some concrete examples from the early part of the book.

Every effort was made to develop the theory into a harmonious unity. The rather detailed Table of Contents and the Table of strict inclusion relations reveal some of the strands of the rather intricate logical network tying the chapters into a whole which we hope to be something more than the sum of its parts.

On occasion a result may seem isolated until its significance manifests itself in relationships given in later chapters. In this regard the Subject Index is essential as it gives cross-sections on specific topics, e.g. a particular null class.

Some subsections, indicated by an asterisk before the heading, are not needed for the understanding of the subsequent parts of the book.

Bibliographical references, summarized in the Author Index, are complete in that the source of every result not due to the authors is explicitly given.

The reader is not expected to have any previous knowledge of classification theory. For general prerequisites a standard Ph. D. curriculum is sufficient. In the few instances where we have made an exception, a precise reference is given to some well-known monograph.

The basic terminology we use is that adopted in Ahlfors-Sario [1].

Although some vague ideas for the book go back two decades, the actual planning, writing, and revising was carried out during the past five years, in particular while the junior author was visiting at UCLA

in 1965–1967. We are deeply grateful to the U.S. Army Research Office–Durham for several grants which made our collaboration possible, and to Drs. J. Dawson and A. S. Galbraith for their magnificent cooperation through the entire course of the work.

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We feel privileged for the inclusion of our book in this distinguished series and we wish to thank Professor B. Eckmann for his interest and encouragement.

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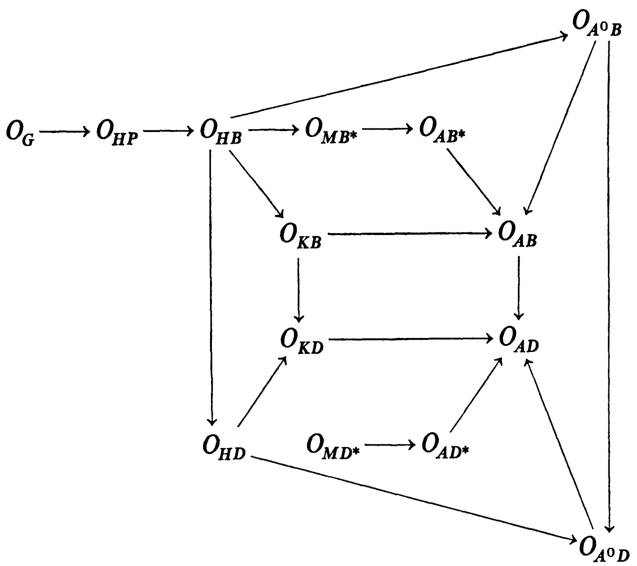

$$
\begin{array}{ccccccccccccccc}
O_G & < & O_{HP} & = & O_{HP}^1 & < & O_{HP}^2 & < & \cdots & < & O_{HP}^{\infty} & < & U_{HP} \cup O_G & & \\
 & & \wedge & & \wedge & & \wedge & & & & \wedge & & \vee & & \\
 & & O_{HB} & = & O_{HB}^1 & < & O_{HB}^2 & < & \cdots & < & O_{HB}^{\infty} & < & U_{HB} \cup O_G & < & O_{AB} \\
 & & \wedge & & \wedge & & \wedge & & & & \wedge & & \wedge & & \wedge \\
 & & O_{HD} & = & O_{HD}^1 & < & O_{HD}^2 & < & \cdots & < & O_{HD}^{\infty} & < & U_{\widetilde{HD}} \cup O_G & < & O_{AD}
\end{array}
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Table of strict inclusion relations

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Chapter II · Other Classes of Analytic Functions

Chapter III · Dirichlet Finite Harmonic Functions

Chapter IV · Other Classes of Harmonic Functions

Chapter V · Functions with Logarithmic Singularities

Chapter VI · Functions with Iversen's Property

Appendix. Higher Dimensions