Contents

Preface v

Chapter 1
The classical planes 1

11 The classical division algebras 4
12 The classical affine planes 25
13 The projective planes over $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ 42
14 The planes over $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ as topological planes 56
15 Geometry of a projective line 65
16 The projective octonion plane $\mathcal{P}_2\mathbb{O}$ 77
17 The collineation group of $\mathcal{P}_2\mathbb{O}$ 88
18 Motion groups of $\mathcal{P}_2\mathbb{O}$ 103

Chapter 2
Background on planes, coordinates and collineations 131

21 Projective and affine planes 131
22 Coordinates, ternary fields 135
23 Collineations 137
24 Lenz–Barlotti types 142
25 Translation planes and quasifields 144

Chapter 3
Geometries on surfaces 148

31 $\mathbb{R}^2$-planes 149
32 Two-dimensional compact projective planes 163
33 Towards classification 178
34 The Moulton planes 185
35 Skew hyperbolic planes 196
36 Skew parabola planes 202
37 Planes over Cartesian fields 205
38 Flexibility, rigidity and related topics 212

Chapter 4
Compact projective planes 216

41 The topology of locally compact planes 217
42 Compact, connected planes 225
43 Ternary fields for compact planes 230
44 Automorphism groups 235

Chapter 5
Algebraic topology of compact, connected planes 244

51 General properties 245
52 Assuming that lines are manifolds 257
53 Conditions implying that lines are manifolds 266
54 Lines are homology manifolds 276
55 Geometric consequences 283

Chapter 6
Homogeneity 309

61 Axial collineations 311
62 Planes admitting a classical motion group 323
63 Transitive groups 339
64 Transitive axial groups 349
65 Groups of large dimension 368
66 Remarks on von Staudt's point of view 370

Chapter 7
Four-dimensional planes 372

71 Automorphism groups 373
72 Characterizing $\mathcal{P}_2\mathbb{C}$ 383
73 Four-dimensional translation planes 393
74 Four-dimensional shift planes 420
75 Analytic planes 445

Chapter 8
Eight- and sixteen-dimensional planes 449

81 Translation planes 452
82 Classification of translation planes 477
83 Stiffness 519
84 Characterizing $\mathcal{P}_2\mathbb{H}$ 535
85 Characterizing $\mathcal{P}_2\mathbb{O}$ 547

86 Hughes planes 555
87 Principles of classification 581

Chapter 9
Appendix: Tools from topology and Lie theory 590

91 Permutation groups 590
92 Topological dimension and remarks on general topology 592
93 Locally compact groups and Lie groups 597
94 Lie groups and their structure 602
95 Linear representations 616
96 Transformation groups 630

Bibliography 643
Notation 679
Index 683