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Logarithmic Potentials with External Fields

With 18 Figures

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To Loretta and Veronika

Preface

In recent years approximation theory and the theory of orthogonal polynomials have witnessed a dramatic increase in the number of solutions of difficult and previously untouchable problems. This is due to the interaction of approximation theoretical techniques with classical potential theory (more precisely, the theory of logarithmic potentials, which is directly related to polynomials and to problems in the plane or on the real line). Most of the applications are based on an extension of classical logarithmic potential theory to the case when there is a weight (external field) present. The list of recent developments is quite impressive and includes: creation of the theory of non-classical orthogonal polynomials with respect to exponential weights; the theory of orthogonal polynomials with respect to general measures with compact support; the theory of incomplete polynomials and their widespread generalizations, and the theory of multipoint Padé approximation. The new approach has produced long sought solutions for many problems; most notably, the Freud problems on the asymptotics of orthogonal polynomials with respect to weights of the form $\exp(-|x|^\alpha)$; the "1/9-th" conjecture on rational approximation of $\exp(x)$; and the problem of the exact asymptotic constant in the rational approximation of $|x|$.

One aim of the present book is to provide a self-contained introduction to the aforementioned "weighted" potential theory as well as to its numerous applications. As a side-product we shall also fully develop the classical theory of logarithmic potentials.

Perhaps the easiest way to describe the main aspects of this work is to use the electrostatic interpretation of the underlying basic extremal problem. We assure the mathematically oriented reader that in what follows we do not use any deep concepts from physics, and do not appeal to anything in our "physical" interpretation that is not intuitively absorbable.

The fundamental electrostatics problem concerns the equilibrium distribution of a unit charge on a conductor. If the conductor is regarded as a compact set E in the complex plane $\mathbf{C}$ and charges repel each other according to an inverse distance law, then in the absence of an external field, equilibrium will be reached when the total energy

$$I(\mu) = \int\int \log \frac{1}{|z-t|} d\mu(z) d\mu(t)$$

is minimal among all possible charge distributions (measures) μ on E having total charge one. There is a unique distribution μ_E supported on E for which this minimal energy is attained; this *equilibrium distribution* μ_E is actually supported on the outer boundary of E, and its logarithmic potential

$$U^{\mu_E}(z) = \int \log \frac{1}{|z-t|} d\mu_E(t)$$

is essentially constant on E. The facts that the support set of μ_E is known and that the potential $U^{\mu_E}(z)$ is essentially constant on this set enable the use of Stieltjes-type inversion formulas to readily determine μ_E.

The distribution μ_E arises in a variety of problems encountered in constructive analysis. For example, it describes the limiting behavior (as $n \to \infty$) of n points on E, the product of whose mutual distances is maximal. These so-called *Fekete points* provide nearly optimal choices for points of polynomial interpolation. In the study of orthogonal polynomials with respect to a large class of (regular) measures on a compact set $E \subset \mathbf{R}$, the equilibrium measure μ_E gives the limiting distribution of the zeros.

The introduction of an external field $Q(z)$ in the electrostatics problem creates some significant differences in the fundamental theory, but opens much wider doors to applications. The problem now becomes that of minimizing the *weighted energy*

$$I_w(\mu) = \int\int \log \frac{1}{|z-t|w(z)w(t)} d\mu(z)d\mu(t) = I(\mu) + 2\int Q d\mu,$$

where the weight $w = e^{-Q}$, and the minimum is again taken over all unit charges μ supported on E.

The external field problem has its origins in the work of C. F. Gauss, and is sometimes referred to as the *Gauss variation problem*. O. Frostman investigated the problem and the Polish school headed by F. Leja made important contributions during the period 1935–1960 that have greatly influenced the present work. A rebirth of interest in the Gauss variational problem occurred in the 1980's when E. A. Rakhmanov and, independently, collaborators H. N. Mhaskar and E. B. Saff used potentials with external fields to study orthogonal polynomials with respect to exponential weights on the real line.

The external field problem is often treated in the literature as an addendum to the classical theory—a generalization for which the similarities with the unweighted case ($Q \equiv 0$) are the main emphasis. On the other hand, this energy problem can be viewed as a special case of the potential theory developed for energy integrals having symmetric, lower-semicontinuous kernels in locally compact spaces. But in this generality many of the unique features of the external field problem, as well as its concrete applications to constructive analysis, remain hidden.

Our goal in writing this book has been to present a self-contained and fairly comprehensive treatment of the Gauss variation problem in the plane, beginning with a review in Chapter 0 on harmonic functions. This is followed by a detailed treatment of Frostman type for the existence and uniqueness of the *extremal measure* μ_w satisfying

$$I_w(\mu_w) = \min\{I_w(\mu) \mid \operatorname{supp}(\mu) \subset E, \quad ||\mu|| = 1\}.$$

Our analysis applies even for unbounded closed sets E, under suitable assumptions on the weight w (or, equivalently, on the external field Q). In this early stage of the development we encounter one of the most glaring differences with the classical (unweighted) electrostatics problem; namely, the support $\mathcal{S}_w$ of the extremal measure μ_w need not coincide with the outer boundary of E and, in fact, can be quite an arbitrary subset of E (depending on w), possibly with positive area. Determining the support set $\mathcal{S}_w$ and its properties are two of the main themes of this work that distinguish it from standard treatments in the literature.

There are several important aspects of the external field problem (and its extension to signed measures) that justify its special attention. The most striking is that it provides a unified approach to several (seemingly different) problems in constructive analysis. These include, among others, the following:

(a) The asymptotic analysis of polynomials orthogonal with respect to a weight function on an *unbounded* interval (e.g., exponential weights of the form $\exp(-|x|^\alpha)$, $\alpha > 0$, on $\mathbf{R}$).
(b) The asymptotic behavior (as $n \to \infty$) of *weighted Fekete points* that maximize the product
$$\prod_{1 \le i < j \le n} |z_i - z_j| w(z_i) w(z_j)$$
among all n-tuples of points $(z_1, \ldots, z_n)$ lying in a closed set E.
(c) The existence and construction of *fast decreasing polynomials*; that is, polynomials $p_n(x)$ of degree n that satisfy for a prescribed nonnegative function $\varphi(x)$ on $[-1, 1]$ the restrictive growth estimates
$$p_n(0) = 1, \quad |p_n(x)| \le \exp(-n\varphi(x)) \text{ for } x \in [-1, 1].$$
(d) The study of *incomplete polynomials* of the form $\Sigma_{k=s}^{n} a_k x^k$ with $s \ge \theta n$ $(\theta > 0)$.
(e) The *numerical conformal mapping* of simply and doubly connected domains onto a disk and annulus, respectively.
(f) A generalization of the *Weierstrass approximation theorem* wherein, for a given weight function w on a closed set E, one seeks to characterize those continuous functions f on E that are uniform limits of weighted polynomials of the form $w^n p_n$, where the power n of the weight is the same as the degree of the polynomial p_n.
(g) The asymptotic behavior of "ray sequences" of *Padé approximants* (interpolating rational functions) to Markov and Stieltjes functions.
(h) The determination of *rates of convergence* of best approximating rational functions to certain classes of functions f (for example, $f(x) = e^{-x}$ on $[0, +\infty)$).

(i) The mathematical modelling of elasticity problems where the shape of the elastic medium is distorted by the insertion of an object under pressure.

In addition, the external field problem provides a rather natural setting for several important concepts in potential theory itself. These include:

(a) Solving simultaneous *Dirichlet's problems*, which arises from the fact that the equilibrium potential $U^{\mu_w}(z)$, with $w = e^{-Q}$, solves this problem (up to a constant) for boundary data $-Q(z)$ on each bounded component of the complement of the support set S_w.
(b) The *balayage* (sweeping) of a measure ν to a compact set E, which is simply given by the extremal measure for the external field $Q(z) = -U^{\nu}(z)$ on E.
(c) The problem of finding the *best Green potential approximation* to a given superharmonic function with respect to an energy norm, which is given by the solution to a Gauss variational problem.
(d) Solving *constrained minimal energy problems* for which one seeks a unit measure λ that minimizes the (unweighted) energy integral for unit measures on E subject to the constraint $\lambda \le \sigma$, where σ is a given positive measure with $\operatorname{supp}(\sigma) = E$ and $||\sigma|| > 1$.

In developing the theory for potentials in the presence of an external field (Chapters I and II), we provide motivations and detailed proofs for many of the basic results from potential theory, such as generalized maximum principles, the Riesz decomposition theorem, the principle of domination, Evans' theorem, etc. These results are presented as they are needed and, as an aid for the reader, we provide a listing of them in the Appendix along with their locations in the text. Wiener's theorem and the Dirichlet problem are also treated in the Appendix.

At the end of each of the main chapters we have included a section entitled "Notes and Historical References," that includes discussion of related results along with citations for many of the theorems presented in the text. There are, however, many new results and proofs that appear here for the first time, such are the ones that are not referenced in the Notes sections.

While our analysis of the weighted energy problem proceeds along the lines of classical potential theory, alternative approaches are being developed, most notably by L. A. Pastur and his collaborators who use random matrix techniques (see Section IV.9). Furthermore, inverse spectral methods have recently been employed by P. Deift, T. Kriecherbauer and K. T-R. McLaughlin to derive more detailed information about the equilibrium distributions for certain smooth fields Q (see the Notes section for Chapter IV).

The theory of weighted potentials in $\mathbf{C}^N$, $N \ge 2$, is still in its infancy relative to the single variable case. To introduce the reader to this vital subject we have included an appendix written by Thomas Bloom that contains generalizations of several theorems in the text to the multidimensional case. This presentation emphasizes the role that the Monge-Ampère operator plays in extending the external field problem to the pluripotential setting.

We are indebted to a large and distinguished cast of students and colleagues who have provided us with valuable feedback on this project. To T. Bloom, A. B. J. Kuijlaars, N. Levenberg, A. L. Levin, D. S. Lubinsky, H. Mhaskar, V. Prokhorov, and H. Stahl we extend our sincere appreciation for their input and encouragement. For their careful reading of the manuscript we especially wish to acknowledge S. Damelin, P. Dragnev, I. Ivanov, P. Simeonov, and Y. Zhou.

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E. B. Saff · V. Totik

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