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Function Theory in the Unit Ball of $\mathbb{C}^n$

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Preface

Around 1970, an abrupt change occurred in the study of holomorphic functions of several complex variables. Sheaves vanished into the background, and attention was focused on integral formulas and on the "hard analysis" problems that could be attacked with them: boundary behavior, complex-tangential phenomena, solutions of the $\bar{\partial}$-problem with control over growth and smoothness, quantitative theorems about zero-varieties, and so on. The present book describes some of these developments in the simple setting of the unit ball of $\mathbb{C}^n$.

There are several reasons for choosing the ball for our principal stage. The ball is the prototype of two important classes of regions that have been studied in depth, namely the strictly pseudoconvex domains and the bounded symmetric ones. The presence of the second structure (i.e., the existence of a transitive group of automorphisms) makes it possible to develop the basic machinery with a minimum of fuss and bother. The principal ideas can be presented quite concretely and explicitly in the ball, and one can quickly arrive at specific theorems of obvious interest. Once one has seen these in this simple context, it should be much easier to learn the more complicated machinery (developed largely by Henkin and his co-workers) that extends them to arbitrary strictly pseudoconvex domains.

In some parts of the book (for instance, in Chapters 14–16) it would, however, have been unnatural to confine our attention exclusively to the ball, and no significant simplifications would have resulted from such a restriction.

Since the Contents lists the topics that are covered, this may be the place to mention some that might have been included but were not:

The fact that the automorphisms of the ball form a Lie group has been totally ignored.

There is no discussion of concepts such as curvature or geodesics with respect to the geometry that has these automorphisms as isometries.

The Heisenberg group is only mentioned in passing, although it is an active field of investigation in which harmonic analysis interacts with several complex variables.

Most of the refined estimates that allow one to control solutions of the $\bar{\partial}$-problem have been omitted. I have included what was needed to present the

Henkin–Skoda theorem that characterizes the zeros of functions of the Nevanlinna class.

Functions of bounded mean oscillation are not mentioned, although they have entered the field of several complex variables and will certainly play an important role there in the future.

To some extent, these omissions are due to considerations of space—I wanted to write a book of reasonable size—but primarily they are of course a matter of personal choice.

As regards prerequisites, they consist of advanced calculus, the basic facts about holomorphic functions of one complex variable, the Lebesgue theory of measure and integration, and a little functional analysis. The existence of Haar measure on the group of unitary matrices is the most sophisticated fact assumed from harmonic analysis. Everything that refers specifically to several complex variables is proved.

I have included a collection of open problems, in the hope that this may be one way to get them solved. Some of these look very simple. The fact that they are still unsolved shows quite clearly that we have barely begun to understand what really goes on in this area of analysis, in spite of the considerable progress that has been made.

I have tried to be as accurate as possible with regard to credits and priorities. The literature grows so rapidly, however, that I may have overlooked some important contributions. If this happened, I offer my sincere apologies to their authors.

Several friends have helped me to learn the material that is presented here—in conversations, by correspondence, and in writing joint papers. Among these, I especially thank Pat Ahern, Frank Forelli, John Fornaess, Alex Nagel, and Lee Stout.

Finally, I take this opportunity to express my appreciation to the National Science Foundation for supporting my work over a period of many years, to the William F. Vilas Trust Estate for one of its Research Professorships, and to the Mathematics Department of the University of Wisconsin for being such a friendly and stimulating place to work in.

Madison, Wisconsin Walter Rudin
March 1980

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List of Symbols and Notations

The numbers that follow the symbols indicate the paragraphs in which their meanings are explained. For example, 10.4.2 means Chapter 10, Section 4, paragraph 2.

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